Applications Of One Type Of Euler-Lagrange Fractional Differential Equation

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Abstract - *In this paper, our work is to present some applications of one type of Euler-Lagrange fractional differential equation with a composition of left fractional derivative of Riemann-Liouville of α order and right fractional derivative of Caputo of α order $,0 < \alpha < 1$. For these applications, we first considered some harmonic oscillators whose equations of motion are expressed in terms of derivatives of non-fractional order and then we transformed these equations into an equation with fractional derivatives. The expanded form of the fractional differential equation is obtained using the finite difference method and the definitions of these two fractional derivatives and then the translation in the form of a matrix equation, is also found. With the examples we have taken, programming in the Matlab language is the method chosen to graphically represent the approximate numerical solutions of this type of equation.*

Key Words: fractional derivatives, Euler-Lagrange differential equation, expanded form, numerical solutions, Matlab Script.

1.INTRODUCTION

The Euler Lagrange equation can be presented in two forms, either in its ordinary form or in fractional form. In its fractional form, the equation is a composition of fractional derivatives.

In order to solve the oscillation of a harmonic pendulum, the Euler Lagrange equation composed of the fractional derivatives of Riemann-Liouville and Caputo of α order was chosen for this work.

1.1 Presentation of the fractional differential equation to be studied

In recent years, fractional calculations and fractional differential equations have interested many scientists to model different problems in the fields of physics, science, etc.

The type of equation that we are going to study, is a composition of left Riemann-Liouville and right Caputo fractional derivatives of α order. It is writing into the following form Eq.1.

$$^{c}D_{1_-}^{\alpha} D_{0^+}^{\alpha} f(t) + rf(t) = 0 \qquad (1)$$

- Where : $t \in [0;1]$;
- f is unknown function to be determined;
- r is a given value;
- $0 < \alpha < 1$
- $^{c}D_{1_-}^{\alpha}, D_{0^+}^{\alpha}$ are respectively the fractional derivatives of α order on the right and on the left of Caputo and of Riemann-Liouville.

1.2 Presentation of the expanded form and matrix notation of equation (1)

Concerning this type of Euler-Lagrange fractional differential equation, we sought the expanded form of this equation, using the method of finite differences of order 1 and the definitions of the two fractional derivatives. For this, we have subdivided the interval $[0;1]$ into N parts, of regular pitch $h = \dfrac{1}{N}$, the points are denoted t_i, $0 \leq i \leq N$, with $t_0 = 0$ and $t_N = 1$.

For $1 \leq i \leq N-1$, we have noted $t_i = ih = \dfrac{i}{N}$ and f_i the approximate solution at each point t_i.

Thus, we obtained the following expanded form:

$$\sum_{k=0}^{N} a_{i,k} f_k = 0 \qquad \text{for } 0 \leq k \leq N, 1 \leq i \leq N-1 \quad (2)$$

Where :

- $a_{i,k} = h^{-2\alpha} \left(\displaystyle\sum_{j=i}^{N} d_{i,j} c_{j,k} \right)$, for $0 \leq k < i$;

- $a_{i,k} = h^{-2\alpha} \left(\displaystyle\sum_{j=i}^{N} d_{i,j} c_{j,k} \right) + r$, for $k = i$;

- $a_{i,k} = h^{-2\alpha} \left(\sum_{j=k}^{N} d_{i,j} c_{j,k} \right)$, for $i+1 \le k \le N$

- $a_{0,0} = 1, a_{N,N} = 1$, the chosen boundary conditions are : $f(0) = f_0 = 0$
  and $f(1) = f_N = 1$.

(3)

So, the matrix form of this equation is as follows:

$$\begin{pmatrix} 1 & 0 & . & . & . & 0 & 0 \\ a_{1,0} & a_{1,1} & . & . & . & a_{1,N-1} & a_{1,N} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{0,N-1} & a_{1,N-1} & . & . & . & a_{N-1,N-1} & a_{N-1,N} \\ 0 & 0 & . & . & . & 0 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ . \\ . \\ . \\ f_{N-1} \\ f_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix}$$ (4)

The approximate solutions to be determined under the Matlab Script are the $f_i, 0 \le i \le N$.

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2. Transformation of an equation of motion of a usual harmonic oscillator into a fractional harmonic oscillator equation

We recall that an equation of motion of a usual, undamped, harmonic oscillator, is an ordinary second-order differential equation of the form:

$$f''(t) + \omega^2 f(t) = 0 \qquad (5)$$

Where ω is called proper pulsation of motion and f is the general solution of this equation.

We can cite some examples of the following harmonic oscillators:

- the horizontal elastic pendulum: an object of mass μ attached to a spring of constant stiffness k in motion around the its equilibrium position and the own pulsation is $\omega = \sqrt{\dfrac{k}{\mu}}$. Its equation of motion is written:

$$x''(t) + \frac{k}{\mu} x(t) = 0 \quad \text{where } x(t) \text{ is the extension of the}$$

spring.

The general solution is of the form: $f(t) = A\sin(\omega t + \varphi)$ where the values of A and φ depend on the chosen boundary conditions.

- the undamped torsion pendulum which performs a rotational movement around its equilibrium position: it is a rotational harmonic oscillator.

The equation of motion is:

$\theta''(t) + \dfrac{c}{J_\Delta} \theta(t) = 0$ where J_Δ is the moment of inertia of

the solid with respect to the axis of rotation and   is the moment of the restoring torque of the torsion wire. The own pulsation is $\omega = \sqrt{\dfrac{c}{J_\Delta}}$.

- A circuit LC which is an electrical harmonic oscillator formed of a perfect coil with inductance L and a capacitor with capacitance C. It is the seat of free, undamped electrical oscillations, of proper pulsation $\omega = \sqrt{\dfrac{1}{LC}}$.

The differential equation governing the variation of the load q in this circuit LC is the following:

- A simple pendulum: an object of mass μ suspended on a wire of length I of negligible mass. This system, in motion, oscillates around its equilibrium position and for low amplitude oscillations $0 < \theta \le 10°$, of proper pulsation $\omega = \sqrt{\dfrac{g}{l}}$, the equation of motion is written:

$$\theta'' + \frac{g}{l} \theta = 0$$

If we transform all these ordinary differential equations of integer order 2, into an equation with compositions of fractional derivatives of order $\alpha, 0 < \alpha < 1$, then we obtain the following type of Euler-Lagrange fractional differential equation:

$$-{}^c D_{1_-}^\alpha D_{0^+}^\alpha f(t) + \omega^2 f(t) = 0 \qquad (6)$$

Or

$${}^c D_{1_-}^\alpha D_{0^+}^\alpha f(t) - \omega^2 f(t) = 0 \qquad (7)$$

This last equation is indeed a fractional Euler-Lagrange differential equation of the form:

$${}^c D_{1_-}^\alpha D_{0^+}^\alpha f(t) + rf(t) = mg(t) \qquad (8)$$

where $r = -\omega^2, g(t) = 0$

3. Programming under Matlab and representations of curves for fractional and non-fractional

Matlab Script is the method of numerical resolution that we have chosen.

```matlab
function f=varnalpha(N,alpha)

% N=input('valeur N=')
% alpha=input('valeur alpha=')
c = zeros(N,N+1);
% Calcul de c
% ************************
for j=1:N
  for k=0:j
    if k ==0
      ta = 1/gamma(2-alpha);
      tb = (1 - alpha)*j^(-alpha);
      tc = (j)^(1-alpha);
      td = (j - 1)^(1-alpha);
      c(j,k+1) = ta * (tb - tc + td);
    elseif  k < j
      ta = 1/gamma(2-alpha);
      tb = (j - k + 1)^(1-alpha);
      tc = (j - k)^(1-alpha);
      td = (j - k - 1)^(1-alpha);
      c(j,k+1) = ta * (tb - 2*tc + td);
    else
      c(j,k+1) = 1/gamma(2-alpha);
    end
  end
end
% Calcul de d
% *******************
d = zeros(N,N);
for i=1 : N-1
  for j= i : N
    if j == i && j ~= N
      d(i,j) = 1/gamma(2-alpha);
    elseif  j < N
      tad = 1/gamma(2-alpha);
      tbd = (j - i + 1)^(1-alpha);
      tcd = (j - i)^(1-alpha);
      tdd = (j - i - 1)^(1-alpha);
      d(i,j) = tad * (tbd - 2*tcd + tdd);
    else
      t1d = (N - i - 1)^(1 - alpha);
      t2d = (N - i)^(1 - alpha);
      d(i,j) = (t1d - t2d)/gamma(2-alpha);
        end
  end
end
% Matrice du système d'équation
% i allant de 1 à N - 1 (profondeur)
% j indice ligne
% k indice colonne
b = 1;
a = zeros(N+1,N+1);
ce = (b / N)^(-2*alpha)
r=-0.75/0.50;
for i = 1 : N - 1
  for k = 0 : N
    temp = 0;

    if k < i
      for j = i : N
        temp = temp + d(i,j)*c(j,k+1);
      end
      a(i+1,k+1) = ce * temp;
    elseif k == i
      for j = i : N
        temp = temp + d(i,j)*c(j,k+1);
      end
      a(i+1,k+1) = ce * temp + r;
    else
      for j = k : N
        temp = temp + d(i,j)*c(j,k+1);
      end
      a(i+1,k+1) = ce * temp;
    end
  end
end

% Remplissage i=1 et i=N+1
% **************************
cc=1;
bb=zeros(N+1,1);
for i = 2 : N
   bb(i,1)=0;
end
a(1,1)=1;
a(N+1,N+1)=1;
bb(1,1)=0;
bb(N+1,1)=1;
f = a\bb;

% calcul f = varnalpha(N,alpha) ; affiche(N,f,color)
figure
grid minor
hold on
xlabel('t')
ylabel('f(t)')
N = 1000;

f0 = varnalpha(N,.5);
affiche(N,f0,'r')

f1 = varnalpha(N,.6);
affiche(N,f1,'g')
f2 = varnalpha(N,.8);
affiche(N,f2,'b')
f3 = varnalpha(N,.9);
affiche(N,f3,'y')
f4=varnalpha(N,1);
affiche(N,f4,'c')

% title('\alpha=0,9')
title('N = 1000')
legend('\alpha=0,5','\alpha=0,6','\alpha=0.8','\alpha=0.9','\alpha=1')
```

We have taken $r = -\dfrac{k}{\mu}$ in the case of the elastic pendulum

for figure 1; and $r = \dfrac{-1}{LC}$ in the case of the circuit LC for

figure 2.

For the following graphical representations, let us take the same parameters for the non-fractional harmonic oscillators, of order $\alpha = 1$ and the fractional harmonic oscillators of order $\alpha, 0 < \alpha < 1$, with the boundary conditions $f(0) = 0, f(1) = 1$ (Fig-3).
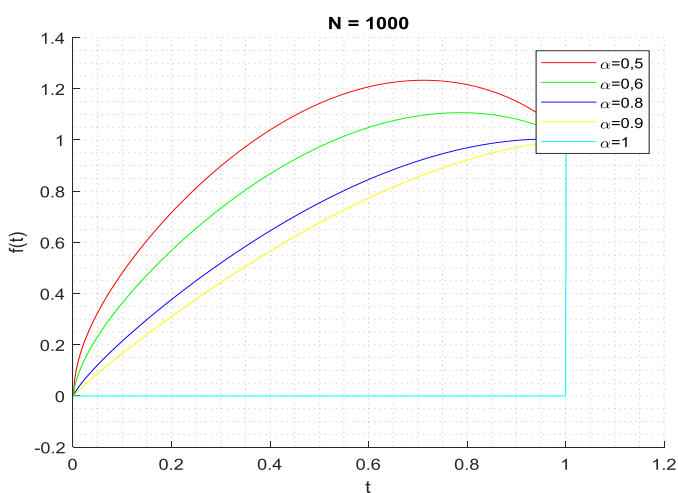

Fig.1 : Numerical solutions for $k = 0.75$ and $\mu = 0.50$ for the case of the horizontal elastic pendulum
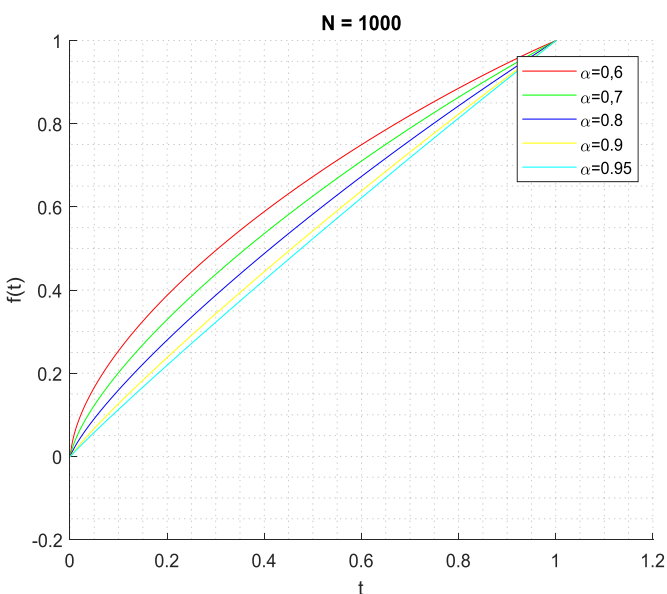

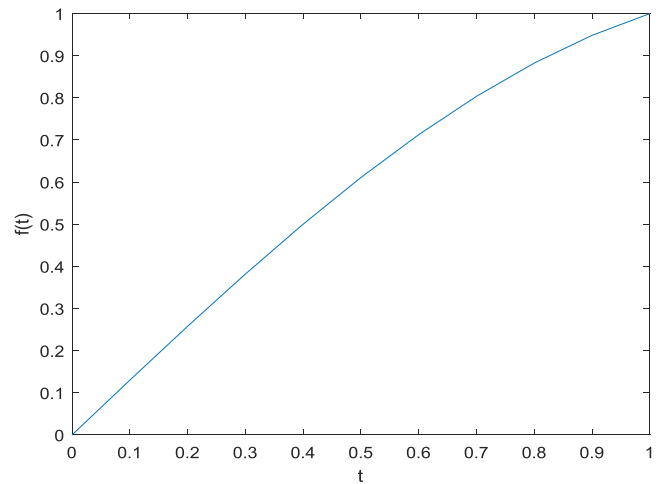**Fig.2 :** Numerical solutions for $L = 0.1$ and $C = 100$ for the case of the circuit LC

Fig.3 : Graphical result

In Fig-3, $f(t) = \dfrac{1}{\sin\left(\sqrt{1.5}\right)} \sin\left(\sqrt{1.5}t\right)$ is the solution for

$\alpha = 1, k = 0.75, \mu = 0.50$.

Using this programming, we were able to represent these curves. We chose $N = 1000$ and we took different fractional α values.

We notice that there is a difference between the graphical representations in the case where $\alpha = 1$, non-fractional order, and the case for the different values of $\alpha, 0 < \alpha < 1$. It seems that the curves get closer when the order α, of the fractional derivative, is close to 1.

4. Conclusion

During an experiment, if there are values of the parameters such that the curve approaches the curve of the fractional case, it is better not to be satisfied with the non-fractional harmonic oscillator but it is necessary to switch to the type of fractional differential equation of Euler-Lagrange that we propose.

Our perspective is to continue our research to find other models of fractional differential equations and then apply them in various fields such as science, engineering, etc.

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