Review on Homotopy Perturbation Transform Method: A Powerful Tool for the Analytical Solution of Linear & Non- Linear Partial Differential Equations

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Abstract – *Linear & nonlinear partial differential equations play very important role in Fluid Dynamics, Magnetohydrodynamics, Hydraulics, Theory of elasticity and so on. Various analytical and numerical methods are available for finding an exact or approximate solution of these equations. In this review paper, one such powerful analytical method, known as Homotopy Perturbation Transform Method (HPTM) is discussed. This method is a combination of good old homotopy perturbation method & Laplace transform technique. The method is not complicated and saves lot of computational work. In non-linear partial differential equation, after using the Laplace Transform, the nonlinear terms in the equation are simplified by using He's polynomials giving an exact or approximate solution.*

Key Words: **Homotopy Perturbation Transform Method, Laplace Transform, He's Polynomials, Non-linear Partial differential equations.**

1.INTRODUCTION

Non-linear phenomena play a vital role in Fluid Mechanics, quantum Physics, Magnetohydrodynamics. There are various analytical & numerical methods available to find its exact or approximate solution. In recent years many researchers have paid attention to solve non-linear partial differential equations by Adomian decomposition method (ADM), homotopy perturbation method (HPM), differential transform method (DTM), and variational iteration method (VIM). These methods require lot of computational work. J. H. He developed the homotopy perturbation method (HPM) by merging the standard homotopy & perturbation technique to solve various physical problems. Further in recent papers Khan and Wu proposed the homotopy perturbation transform method (HPTM) for solving non-linear equations. This method is a combination of homotopy perturbation method (HPM), Laplace transform technique and He's polynomials. The HPTM provides the solution in rapid convergent series which gives an exact solution in the closed form.

J.H. He (1999) [5] described the fundamentals of Homotopy Perturbation method. He had showed by taking some examples that the homotopy perturbation technique does not depend upon a small parameter in the equation. In topology, a homotopy is constructed with an embedding parameter
$q\in [0,1]$ which is considered as a "small parameter". It was demonstrated through his method that the approximations obtained by the proposed method are uniformly valid not only for both small parameters as well as for very large parameters. It is considered as a promising and evolving method. Not only it has importance in solving mathematical equations, it can be applied to other branches of modern sciences.

Swielam and Khadar (2008) [11] obtained exact solution of some coupled non-linear differential equations using homotopy perturbation method using Laplace transform and Pade approximation. Two test examples were given; the coupled nonlinear system of Burger equations and the coupled nonlinear system in one dimensional thermoelasticity. The results obtained ensured that this modification was capable of solving a large number of nonlinear differential equations that had wide application in physics and engineering as well apart from mathematical equations.

The HPM does not need small parameters so that the limitations and non-physical assumptions present in the previous method are eliminated. Therefore, this modification of HPM has widely been used in solving nonlinear problems to overcome the shortcoming of other methods such as Adomian decomposition method.

The use of He's polynomials in the nonlinear term was first introduced by Ghorbani (2009) [2] The Adomian decomposition method (ADM) is widely used by many researchers in approximate calculation. The main difficulty of ADM is to calculate Adomian polynomials, the procedure is very complex and many iterations increase the computational work. In order to overcome the demerit, this paper suggests an alternative approach to Adomian method, instead of Adomian polynomials, He polynomials are introduced based on homotopy perturbation method. The solution procedure becomes easier, simpler, and more straightforward.

homotopy perturbation method and He polynomials can completely replace the Adomian method and Adomian polynomials.

Yasir Khan and Wu Q. (2010) [12] have used HPTM for solving non-linear advection equations. For homogeneous & non homogeneous advection equations, an exact solution in the closed form is obtained. It is verified that HPTM is very effective & easy analytic method which is a clear advantage over Adomian Decomposition Method (ADM).

Gupta and Gupta (2011) [3] had showed applications of homotopy perturbation transform method for solving initial boundary value problem of variable coefficients. This method proved to be very effective for higher dimensional initial boundary value problems with variable coefficients. Four problems have been discussed to prove the usefulness of PTM.

Madani et al (2011) [7] applied the technique in mathematical modelling. One dimensional non homogeneous partial differential equation with variable coefficients have been solved by using HPTM. Since the method is very easy to apply & the use of Laplace Transforms is to overcome the deficiency caused by unsatisfied conditions in other classical methods such as HPM, VIM and ADM. Approximate solutions obtained by HPTM were compared with HPM and finite element method. Also, HPM, VIM and ADM can be used to solve the non-homogeneous variable coefficient PDE with accurate approximation only for small range & boundary conditions in one dimension are satisfied via these methods but unsatisfied conditions do not have any role in final results. To overcome these deficiencies, HPTM is becoming useful and efficient technique. Authors have obtained accurate results in a wide range with one or two iterations. Time dependent non homogeneous PDE are solved by HPTM. Also, comparison between HPM and HPTM proves the efficiency of HPTM.

Jagdev Singh et al (2012) [4]-In this paper, the homotopy perturbation transform method (HPTM) has been applied to obtain the solution of the linear and nonlinear Klein-Gordon equations. With initial conditions. The method is reliable and easy to use. The results show that the homotopy perturbation transform method is powerful and efficient technique in finding exact and approximate solutions for nonlinear differential equations. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. The fact that the HPTM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method. In conclusion, the HPTM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

Devendra Kumar et al (2013) [1] have applied the homotopy perturbation transform method (HPTM) to obtain the solution of linear and nonlinear Schrodinger equations. All the equations give exact solution in closed form. Authors have verified that HPTM is a powerful and reliable technique for linear and non-linear Schrodinger's equation.

Mansi and Patel (2016) [6] further showed the applications of HPTM for the phenomenon of

im-bibation in two immiscible fluid flow through porous media. The Homotopy perturbation trans- form method (HPTM) was applied using He's Polynomial for finding the analytical solution of porous medium equation. Authors have discussed analytically the phenomenon of imbibition in two immiscible fluid flow through porous media. In this paper, we apply the Homotopy perturbation transform method (HPTM) using He's Polynomial for finding the analytical solution of porous medium equation. The HPTM method is a combination of Laplace Transform method and the Homotopy perturbation method. This method is very efficient, simple and can be applied for other non - linear problems also.

In this paper, the solutions for fluid flow through porous medium or nonlinear diffusion equation obtained with different initial conditions by homotopy perturbation transform method. From this method we can conclude that the nonlinear problems have the desired solution. From the graphs we can conclude that the saturation is in the form of exponential form in the first case and in linear form in the second case.

U. Filobello-Nino et al (2016) [13] have proposed this method to solve nonlinear differential equation with Dirichlet, Neumann and mixed boundary conditions. Comparison is made between the figures of exact and approximate solutions, and it is seen that they are of high accuracy.

This paper is very well organized by first introducing the basic idea of HPM and Laplace Transform. Then HPTM is introduced. Three case studies as the application of HPTM are discussed. One is Gelfand's (an ordinary differential) equation for combustible gas dynamics with Dirichlet's boundary condition. Second is an ordinary differential equation of temperature distribution in a uniformly thick rectangular fin radiation to free space with non-linearity of high order involving mixed boundary conditions. Third is an ordinary nonlinear differential equation with Neumann boundary conditions. Comparison between numerical and HPTM solution is shown through graphs. HPTM is proved to be extremely efficient and useful in practical applications.

Mubashra Saleem et al (2017) [10] demonstrated that, for solving nonlinear equations, a combination form of the Laplace transform method along with the Homotopy perturbation method was used. Pro- posed method solves nonlinear problems without using Adomians polynomials can be thought as a vibrant advantage of this algorithm over the Adomian decomposition method.

In this paper, for solving nonlinear equations a combined form of the Laplace transform method along with the Homotopy perturbation method is used. This method is called the Homotopy perturbation transform method (HPTM). He's polynomials can be used to handle the nonlinear terms easily. This method avoids the round-off errors and finds the solution without any discretization or restrictive assumptions. Proposed method solves nonlinear problems without using Adomian's polynomials can be thought as a vibrant advantage of this algorithm over the decomposition method.

In this paper, comparison of homotopy perturbation method (HPM) and homotopy perturbation transform method (HPTM) is effectively shown by Mohammad Elbadri (2018) [8]. Authors have claimed that homotopy perturbation transform method is very fast convergent to the solution of the partial differential equation. For illustration and more explanation of the idea, some examples are provided through which it is shown that most of the inhomogeneous problems give exact solution by HPTM whereas HPM gives an approximate solution to these problems.

Mohamed Jleli et al (2020) [9] have successfully applied HPTM for solving multidimensional time fractional partial differential equations involving the recent introduced Yang-Abdel-Aty-Cattani fractional derivative. The presented examples show that this approach is very powerful and efficient for finding approximate solutions for wide classes of problems arising in science and technology.

2. Methodology:

To illustrate the basic idea for HPTM, we consider a general non-linear partial differential equation with the initial conditions of the form

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \quad (1)$$

with initial condition

$u(x,0) = h(x)$ and $u_t(x,0) = f(x)$

where $D = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represents the general non-linear differential operator and $g(x,t)$ is the source term. Taking Laplace transform of both sides of (1)

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \quad (2)$$

using differentiation property of Laplace transform

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2}L[Ru(x,t)] + \frac{1}{s^2}L[g(x,t)] - \frac{1}{s^2}L[Nu(x,t)] \quad (3)$$

Taking Laplace inverse on both sides of (3)

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2}L(Ru(x,t) + Nu(x,t))\right] \quad (4)$$

where $G(x,t)$ represents the term arising from the source term with prescribed initial conditions

Now by applying homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} q^n u_n(x,t) \quad (5)$$

where q is the embedding parameter &
$q \in [0,1]$,
the non-linear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} q^n H_n(u) \quad (6)$$

for some He's polynomials $H_n(u)$ that are given by

$$H_n(u_0, u_1 \dots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial q^n}\left[N \sum_{i=0}^{\infty} q^i u_i\right]_{q=0}$$

n=0,1,2……

Substituting equation (5) and (6) in (4) we get

$$\sum_{n=0}^{\infty} q^n u_n(x,t) = G(x,t) - q\left\{L^{-1}\left[\frac{1}{s^2}L\left(R\sum_{n=0}^{\infty} q^n u_n(x,t) + \sum q^n H_n(u)\right)\right]\right\} \quad (7)$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials.

Comparing the co-efficient of like powers of q, the following approximations are obtained.

$$q^0 : u_o(x,t) = G(x,t)$$

$$q^1 : u_1(x,t) = -\frac{1}{s^2}L[Ru_0(x,t) + H_0(u)]$$

$$q^2 : u_2(x,t) = -\frac{1}{s^2}L[Ru_1(x,t) + H_1(u)]$$

$$q^3 : u_3(x,t) = -\frac{1}{s^2}L[Ru_2(x,t) + H_2(u)]$$

The best approximation for the solution is

$$u = u_0 + u_1 + u_2 + \dots\dots$$

3. Application:

Ex. (1) Consider non-linear advection equation [3]

$u_t + uu_x = 0$ with initial condition $u(x,0) = -x$.

Taking Laplace transform on both sides & applying initial condition

$$u(x,s) = -\frac{x}{s} - \frac{1}{s}L(uu_x),$$

Taking inverse Laplace transform

$$u(x,t) = -x - L^{-1}\left[\frac{1}{s}L(uu_x)\right],$$

Now applying Homotopy Perturbation method,

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = -x - p\left(L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right),$$

where $H_n(u)$ are He's polynomials representing nonlinear terms

$$H_0(u) = u_0 u_{0x}$$

$$H_1(u) = u_0 u_{1x} + u_1 u_{0x}$$

$$H_2(u) = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}$$

……..........................

Comparing the coefficients of like powers of p

$$p^0 : u_0(x,t) = -x$$

$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s}L[H_0(u)]\right] = -xt$$

$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s}L[H_1(u)]\right] = -xt^2$$

Similarly

$$p^3 : u_3(x,t) = -xt^3$$

$$p^4 : u_4(x,t) = -xt^4$$

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So that the solution becomes

$$u(x,t) = u_0 + u_1 + u_2 + \cdots..$$

$$u(x,t) = -x - xt - xt^2 - xt^3 - xt^4 - \cdots \dots..$$

$$u(x,t) = -x(1 + t + t^2 + t^3 + \cdots)$$

series form

$$u(x,t) = -\frac{x}{1-t}$$

closed form

As an exact solution

Ex. (2) Consider non-linear Schrodinger's equation [1]

$iu_t + u_{xx} + 2|u|^2 u = 0$ with initial condition $u(x,0) = e^{ix}$

Applying Laplace Transform on both sides of (1) we get

L [u (x, t)] = $\frac{e^{ix}}{s} + \frac{1}{s} iL[u_{xx} + 2u^2\bar{u}]$ where $u^2\bar{u} = |u|^2 u$

& $\bar{u}$ is the conjugate of u.

The inverse of Laplace transform implies that

$$u(x,t) = e^{ix} + L^{-1}\{\frac{1}{s} iL[u_{xx} + 2u^2\bar{u}]\}$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{ix} + p\left(L^{-1}\left[\frac{1}{s} iL\left[\sum_{n=0}^{\infty} p^n u_n(x,t)_{xx} + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right)$$

Where $H_n(u)$ are He's Polynomials representing nonlinear terms.

The 1st few components of He's polynomials are

$$H_0(u) = 2u_0{}^2\overline{u_0}$$

$$H_1(u) = 2(u_0{}^2\overline{u_1} + 2u_1 u_0\overline{u_0})$$

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Comparing the coefficients of like powers of p, we have

$$p^0 : u_0(x,t) = e^{ix}$$

$$p^1 : u_1(x,t) = L^{-1}\left\{\frac{1}{s} iL(u_0)_{xx} + H_0(u)]\right\} = ite^{ix}$$

$$p^2 : u_2(x,t) = L^{-1}\left\{\frac{1}{s} iL(u_1)_{xx} + H_1(u)]\right\} = \frac{(it)^2}{2!} e^{ix}$$

Similarly

$$p^3 : u_3(x,t) = \frac{(it)^3}{3!} e^{ix}$$

………… ………………

So that the solution becomes

$$u(x,t) = u_0 + u_1 + u_2 + \cdots ..$$

$$= e^{ix}(1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots ...)$$

In the closed form

$u(x,t) = e^{i(x+t)}$ is an exact solution.

4. CONCLUSION

A review on HPTM will help the researchers to use this as an effective and promising analytical method for linear and nonlinear partial differential equation. In the recent research in nonlinear sciences, many authors have used Adomian Decomposition method for nonlinear PDEs, which needs lot of computational work. Whereas HPTM provides an exact or approximate solution in minimum steps without any discretization or restrictive assumptions. Also, the round-off errors are avoided in this method.

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