How Brilliant Simple – Seemingly Naïve - Cognitive Research Approaches can Succinctly Lead to Subtle Results

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Abstract- *It is a natural expectation and a normal state of affairs that most academicians and a large number of researchers spend quite a time, strain, endeavour or even strive to excel themselves in pursuit of ingenuity, excellence and supremacy. A lot of time and energy is saved if a novel idea comes into play to make a real breakthrough and achieve the desired objective. In this paper we provide two methods targeting the square root of a number. The first method is obtained via a very simple, neat and elegant derivation approach. The theme and elegance of the derivation emanate simply from a rearrangement of the function in question into another equivalent form and an associated trivially simple identity. It deserves to be noted, however, that the derivation process neither involves nor refers to any function derivatives. Despite the unique style and methodology employed in the development and derivation process, the final formula obtained turns out to be an exact facsimile of the Classical Newton Raphson Iteration Formula – the wheel is re-invented!! The second method – claimed new, unless shown otherwise - is obtained via applying the Newton Raphson to a function equivalent to the function in hand.*

Keywords – *naïve; cognitive; succinct ;subtle; strain ;* endeavour ; strive ; ingenuity; supremacy; breakthrough ; identity; iteration; iteration formula; function derivative.

I.INTRODUCTION

Given a function f(x), a usual intention is to find a number (x^* say) such that:-

$$f(x^*) = 0 \qquad (1)$$

Such an x^* is called a zero of f(x).

Starting with x_0 , a rough estimate to the required zero x^* , subsequent better iterates are obtained using the Newton-Raphson Iteration Formula [1] in the form:-

$$x_{n+1} = x_n - f(x_n) / f'(x_n) \qquad (2)$$

Formula (2) - the Newton-Raphson Iteration Formula - applied to the function

$$f(x) = x^2 - A \quad , \quad A > 0 \qquad (3)$$

reduces – after some simplification - to:

$$x_{n+1} = ½ (x_n + A / x_n) \qquad (4)$$

II.DERIVATION OF THE FIRST METHOD

In this paper - quite independently and following a completely different approach – we derive – simply and neatly- a formula targeting the square root of a number. The simplicity and neatness of the derivation of the method emanate from the function addressed and an associated trivial identity – as will be evident.

Further the derivation process does not involve nor refer to any function derivatives.

Fortunately or unfortunately we found - to our dismay and satisfaction – that we have reinvented the wheel!! The derived formula is a reassertion of the classical Newton Raphson's iteration formula!!

Now equating to zero the function in Eq. (3) above, the equation below is obtained:-

$$f(x) = x^2 - A = 0 \qquad (5)$$

On rearranging Eq. (5), we obtain the equivalent form

$$X = A / x \qquad (6)$$

Now adding the trivial identity x = x to both sides of Eq.(6) and dividing the result by 2, we obtain:-

$$x = ½ (x + A / x) \qquad (7)$$

Now, let us rewrite Eq. (7) as

$$x = g(x) , \text{ where } g(x) = ½ (x + A / x) \qquad (7)'$$

Eq. (7)' reduces in iteration format to

$$x_{n+1} = g(x_n) , \text{ where } g(x) = ½ (x + A / x) \qquad (8)$$

Needless to say that Eq. (8) is exactly identical to Eq.(4), emanating from applying the Newton Raphson Formula - NRF- to the quadratic case. Hence the result!!

The function g(x) – in (7)' above - is known as the iteration function and should satisfy a certain criterion to secure the convergence of the iteration process in (8).

Within the context of iteration, the iteration process in (8) is guaranteed to converge if $|g'(x)| < 1$, near the required zero.

Fortunately in this case: $|g'(x)| = |½.(1 - A/x^2)| \approx 0$ (< 1) ,when $x^2 - A \approx 0$.
Hence the iteration given by Eq.(8) converges.

To illustrate the convergence of the iteration process and to appreciate the speed of such a convergence, let us embark on the examples below:-

Example # 1: $f(x) = x^2 - 10 = 0$, whose solution is $\sqrt{10}$.

n	x_n	$g(x_n) = ½.(x_n + 10 / x_n)$	$(x_n)^2$
0	3	3.16 67	9
1	3.1667	3.1622 8	10.028
2	3.16228	3.162277660	10.000015
3	3.16227766	3.162277660168	9.9999999989
4	3.162277660168	3.162277660168	9.9999999999976

Example # 2: $f(x) = x^2 - 2 = 0$, whose solution is $\sqrt{2}$.

n	x_n	$g(x_n) = ½(x_n + 2/ x_n)$	x_n^2
0	1	1.5	1
1	1.5	1.41 66	2.25
2	1.4166	1.41421 56	2.0068
3	1.4142156	1.41421356	2.00000576
4	1.41421356	1.41421356237	1.99999999329
5	1.41421356237	1.41421356237	2

Example # 3: $f(x) = x^2 - 72 = 0$, whose solution is $\sqrt{72}$.

n	x_n	$g(x_n) = ½(x_n + 2/ x_n)$	x_n^2
0	8	8.5	64
1	8.5	8.48529	72.25
2	8.48529	8.48528137	72.000136
3	8.48528137	8.48528137423857	7.199999993
4	8.48528137423857		72

III.DERIVATION OF THE SECOND METHOD

Consider the problem of finding a zero of a function

$f(x) = x^2 - A = 0$ (1) Rewriting

Eq. (1) in the equivalent form

$F(x) = x - A/x = 0$ (2)

F(x) and f(x) have the same zeros.

Now taking x_0 as a rough estimate to a zero x^* of F(x) , then applying the NRF, subsequent better estimates may be obtained in the form :-

$$x_{n+1} = x_n - F(x_n) / F'(x_n) \qquad (3)$$

which gives Eq.(4) below

$$x_{n+1} = x_n - (x_n - A/x_n) / (1 + A/(x_n)^2) \quad (4)$$

After some elementary simplification Eq. (4) reduces to :-

$$x_{n+1} = [2T. (1 + T)^{-1}]. x_n , \text{ where } T = A/(x_n)^2 \qquad (5)$$

Unless – claimed otherwise - we will call this formula: Nourein's Formula !!
To see how Nourein's Formula behaves, let us follow the footsteps of our previous 1st method.

Example #1 $F(x) = x - 10/x = 0$, whose solution is $\sqrt{10}$

n	x_n	$T = 10 / (x_n)^2$	$[2T(1+T)^{-1}] . x_n$
0	3	1.1111	3.1579
1	3.1 579	1.00277	3.16227
2	3.16227	1.0000048447	3.162277660
3	3.162277660	1.000000000005868	3.162277660

Example #2 $F(x) = x - 2/x = 0$, whose solution is $\sqrt{2}$

n	x_n	$T = 2 / (x_n)^2$	$[2T(1+T)^{-1}] . x_n$
0	1	2	1.3333
1	1.3333	1.125056	1.41 176
2	1.41 176	1.0034789	1.41421 142
3	1.41421 142	1.0000302978	1.41421356237
4	1.41421356237	1.000000000004377	1.41421356237

Example # 3: $F(x) = x - 72/x = 0$, whose solution is $\sqrt{72}$.

n	x_n	$T = 72/(x_n)^2$	$2T(1 + T)^{-1} . x_n$
0	8	1.125	8.47
1	8.47	1.0036	8.4852
2	8.4852	1.00001918	8.48528137
3	8.48528137	1.000000000999	8.48528137423857
4	8.48528137423857		

IV.CONCLUSION

At first glance, the iteration function of the 2nd method looks seemingly relatively complex.

Looking at the results of the two methods: as iterations proceed, we observe too close a likeness of the results and the accuracy obtained per iteration - as is manifested by the number of significant digits preserved – shown underlined!.

We felt over concerned to look for an explanation of such a close similarity of the results.

On a closer scrutiny, critical investigation and analysis of the underlying iteration functions : $g_1(x_n)$, $g_2(x_n)$ of the 1st and 2nd methods , respectively, we found the following:-

1st method :- $x_{n+1} = g_1(x_n) = ½. x_n (1+ A/(x_n)^2) = \{½. (1+ T)\}.x_n = \mu_1.x_n$
2nd method :- $x_{n+1} = g_2(x_n) = \{2T(1 + T)^{-1}\} . x_n = \mu_2.x_n$, with the obvious notation.

The identity $(1 + T)^2 = (1 - T)^2 + 4T \rightarrow (1 + T) = 4 T (1 + T)^{-1}$ as $T \rightarrow 1$
$\rightarrow ½.(1 + T) = 2 T (1 + T)^{-1} \rightarrow \mu_1 = \mu_2$.

Now as iteration proceeds, $n \rightarrow \infty$, x_{n+1} , $x_n \rightarrow x^*$ and $T = A/(x_n)^2 \rightarrow A/(x^*)^2 = 1$.
Hence $\mu_1 \rightarrow \mu_2$, justifying the results, [Exemplified below].

Example # 4 $f(x) = x^2 - 2$, whose solution is $\sqrt{2}$

n	x_n	$\mu_1 = ½.(1 + T)$	$\mu_2 = 2 T (1 + T)^{-1}$	μ_1 / μ_2
0	1	1.3333	1.5	.8886
1	1.3333	1.05907	1.06278	.996509
2	1.4176	.997608	.997614	.999994
3	1.4142	1.00000959	1.00000959	1

Example # 5 $f(x) = x^2 - 10$, whose solution is $\sqrt{10}$

n	x_n	$\mu_1 = ½.(1 + T)$	$\mu_2 = 2 T (1 + T)^{-1}$	μ_1 / μ_2
0	3	1.0526	1.0556	.997158
1	3.1579	1.001385	1.001387	.999998
2	3.16227	1.000002422	1.000002422	1

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