Three - Point Boundary Value Problems associated with a System of Generalized Matrix Differential Equations

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Abstract: In this paper, we establish existence and uniqueness of solutions to generalized Three - point boundary value problems in the space of functions of bounded variation using the concept of generalized fundamental matrix given in [5]. The existing results on classical Three - point boundary value problems will become a particular case of our results. The properties of the generalized Green's matrix are presented in a separate theorem. These results have been generalized to Kronecker product linear systems and existence and uniqueness criteria are also presented.

Key Words: Generalized Fundamental matrix, Generalized Green's matrix, Kronecker product of matrices, Existence and uniqueness, Three Point Boundary Value Problems.

AMS (MOS) classification: 34B15, 34B14, 93B05, 93B07.

1.INTRODUCTION:

In this paper, we shall be concerned with the existence and uniqueness of solutions to the generalized three - point boundary value problem.

$$dy = d[A]y+dg, \quad t \in [a, c] \tag{1.1}$$

$$M\,y(a) + N\,y(b) + R\,y(c) = 0, \tag{1.2}$$

where $A \in BV^{n\times n}$, $g \in BV^n$, $y \in BV^n$, M,N and R are all constant square matrices of order n and all scalars are assumed to be real. Throughout the paper [a, c] is a closed subinterval of R, BV^n, $BV^{n\times n}$ denote the Banach space of functions of bounded variation on [a,c] taking values in R, R^n and $R^{n\times n}$respectively. The theory of generalized linear first order system of differential equations in the space of functions of bounded variation isfirst initiated by Kurzurell [3], Halnany and more [2]. In the year 1999 ValeriuPrepelita [5], gave two methods for the calculus of the fundamental matrix for the linear differential equation on the Banach space of functions of Bounded variation. Recently notable researchers like Kasi Viswanadh V. Kanuri, K.N Murty, Yan Wu, Divya L. Nethi, Bhagavathula S., R. Suryanarayana et.al [6-13] established existence and uniqueness of three-point boundary value problems. Kasi Viswanadhet. al established $(\Phi \otimes \Psi)$bounded solutions of Kronecker product sytems. Solutions of Kronecker product initial value problems is initiated in [11]. The work by Yan Wu, Divya L. N, K.N Murty in [12] established existence and uniqueness criteria for Fuzzy difference systems needs a special mention as it is useful in agricultural science and enhance the future of aeronautical scienceVariation of parameters formula established by Kasi Viswanadhet. al[8,9] paved a new area of research on Kronecker Product Linear System of first order differential equations. A new approach to the study of Linear systems are also established in a recent paper by Kasi Viswanadh et. al in [7]. The results established in [6] unifies both continuous and discrete systems in a single frame work..We make use of the calculus of the fundamental matrix of the homogeneous system associated with (1.1) and establish existence and uniqueness criteria for the generalized boundary value problem (1.1) satisfying (1.2).

The homogeneous system corresponding to the non-homogenous system (1.1) is given by

$$dy=d[A]y. \tag{1.3}$$

We define

$$\Delta^+A(t)=A(t^+)-A(t)$$
$$\text{and} \quad \Delta^-A(t)=A(t)-A(t^-).$$

If A is continuous on the left, then $\Delta^-A(t)=0_n$, a null matrix of order n and hence $\det\left(I-\Delta^-A(t)\right)=\det I\neq 0$. On the other hand if A is right continuous then $\Delta^+A(t)=0_n$ and hence $\det\left(I+\Delta^+A(t)\right)=\det I\neq 0$. If the matrix valued function A $\in BV^{n\times n}$ has the properties that $\det\left(I+\Delta^+A(t)\right)\neq 0$ and $\det\left(I-\Delta^-A(t)\right)\neq 0$, then there exists a generalized fundamental matrix $\Phi(t,s):[a,b]\times[a,b]\to R^{n\times n}$ given by [5]

$$\Phi(t,s)=I+\int_s^t d\left[A(r)\right]\Phi(t,s) \tag{1.4}$$

for all t,s$\in$ [a,c]. Note that $\Phi(t,t)=\Phi(s,s)=I$. We further note that

$$\Phi(t^+,s)=\left[I+\Delta^+\left(A(t)\right)\right]\Phi(t,s)$$

and

$$\Phi(t^-,s)=\left[I-\Delta^-\left(A(t)\right)\right]\Phi(t,s).$$

Theorem 1.1. If A $[a_{ij}(t)]\in BV^{n\times n}$ is continuous on the left of [a,b] and satisfies the conditions

$$\det\left[I+\Delta^+A(t)\right]\neq 0, \quad \forall t\in[a,c]$$
$$\text{and } \det\left[I-\Delta^-A(t)\right]\neq 0, \quad \forall t\in[a,c]$$

then for any $t_0\in[a,c]$, $x_0\in R^n$, $g\in G^n$, any solution of the initial value problem :

$$dy=d[A]\,y+dg \tag{1.5}$$
$$y(t_0)=y_0, \tag{1.6}$$

is given by

$$y(t)=\Phi(t,t_0)y_0+g(t)-g(t_0)-\int_{t_0}^t d_s\left[\Phi(t,s)\right]\left(g(s)-g(t_0)\right), \qquad t\in[a,c] \text{ and } y\in G^n.$$

The solution of a non-homogeneous boundary value problem (1.1) satisfying (1.2) is not itself a topic of great importance. However, we present the solution in the form of an integral transform. This transform, characterized by the Kernal (Green's matrix), isof considerable significance. In the next section, we establish existence and uniqueness of solution of three - point boundary value problem (1.1), satisfying (1.2). The results on classical three - point boundary value problem will become a particular case of our results on a system of generalized boundary value problems on the space of functions of bounded variation.

2. MAIN RESULTS:

In this section, we prove our main result on existence and uniqueness of solutions of the generalized boundary value problem (1.1) satisfying (1.2) on the space of functions of bounded variation. Before, we prove our main result we present the following analysis

Definition 2.1. If $\Phi(t,s)$ is any fundamental matrix of the generalized linear differential system (1.1), then the generalized matrix D defined by

D = $M\Phi(a,s)+N\Phi(b,s)+R\Phi(c,s)$ (2.1)

for every fixed $s\in[a,c]$ is called a characteristic matrix for the generalized boundary value problem (1.1) satisfying (1.2).

It may be noted that the given boundary value problem is not essentially altered if the end point condition (1.2) replaced by

$$BMy(a)+BNY(b)+BRY(c)=0,$$

where B is any constant $(n\times n)$ non - singular matrix. Moreover, if Φ is any fundamental matrix for the equation (1.3), then so is ΦC, where C is any constant non-singular $(n\times n)$ matrix. This implies BDC is a characteristic matrix. A characteristic matrix for a generalized boundary value problem is, therefore, for from unique. It is quite clear, however, that all characteristic matrices for a given boundary value problem, have the same rank.

Definition 2.2. The dimension of the solution space of the boundary value problem is the index of compatibility of the generalized boundary value problem. A generalized boundary value problem (1.1) satisfying (1.2) is said to be incompatible if its index of compatibility is zero.

It may be noted that the solution of the homogeneous boundary value problem form a vector space.

Theorem 2.1. If the generalized boundary value problem

dy = d [A] y (2.2)

M Y(a) + N y(b) + R y(c) = 0

has a generalized characteristic matrix of rankr, then its index of compatibility is n-r.

Proof. Any solution of the homogeneous system (1.3) is given by $\Phi(t,t_0)x_0$, where $\Phi(t,s)$ is the generalized fundamental matrix and is given by

$$\Phi(t,s)=I+\int_x^t d[A(r)]\,\Phi(t,s)$$

fort,$s\in[a,c]$. Substituting the general form of the solution in the homogenous boundary conditions, we get $Dx_0=0$. If the rank of the matrix D is r, then $Dx_0=0$ has (n-r) linearly independent solutions for the vector x_0. These solutions yield (n-r) linearly independent solutions for the boundary value (2.2). Hence, the index of compatibility of the boundary value problem is (n-r). This is true for any fixed $x_0\in[a,c]$. This completes the proof of the theorem.

We now, turn our attention to the boundary value problem (1.1) satisfying (1.2). We first assume that the homogenous boundary value problem is incompatible. This restriction only ensures that the characteristics matrix D is nonsingular for each $t\in[a,c]$.

Theorem 2.2. Suppose that the homogenous boundary value problem (1.3) satisfying (1.2) is in-compatible. Then there exists a unique solution of the non-homogeneous boundary value problem (1.1) satisfying (1.2) is given by

$$y(t)=\int_a^c G(t,s)[g(s)-g(a)]ds+D^{-1}[g(c)-g(a)]+D^{-1}[g(b)-g(a)]+g(t)-g(a), \quad (2.3)$$

where G(t,s) is the Green's matrix for the homogeneous boundary value problem and is given by

$$\begin{array}{c} G(t,s) \\ t\in[a,b] \end{array} = \begin{cases} d_s[\Phi(t,s)]\, D^{-1}M\, d_s[\Phi(a,s)] & a \le s \le t \le b \le c \\ -d_s[\Phi(t,s)]D^{-1}[N\, d_s[\Phi(b,s)] + R\, d_s[\Phi(c,s)]] & a \le t \le s \le b \le c \\ -d_s[\Phi(t,s)]\, D^{-1}\, d_s\, [\Phi(c,s)] & a \le t < b \le s \le c \end{cases} . \tag{2.4}$$

and

$$\begin{array}{c} G(t,s) \\ t\in[b,c] \end{array} = \begin{cases} d_s[\Phi(t,s)][I - D^{-1}R\, d_s[\Phi(c,s)]] & a < s < t < b \le c \\ -d_s[\Phi(t,s)] - d_s\Phi(t,s)\, D^{-1}[\, Nd_s[\Phi(b,s)] + R\, d_s[\Phi(c,s)]] & a < b \le t < s < c \\ -d_s[\Phi(t,s)]\, D^{-1}\, R\, d_s[\Phi(c,s)] & a \le s < b \le t \le c \end{cases} . \tag{2.5}$$

Proof. Any solution y(t) of the generalized non-homogeneous system (1.1) is given by

$$y(t) = \Phi(t,s)C + g(t) - g(a) + \int_a^t d_s[\Phi(t,s)]\,[g(s) - g(a)] \qquad t \in [a,c] \tag{2.6}$$

substituting the general form of (2.2) in the boundary condition matrix (1.3), we get

$$[M\Phi(a,s) + N\Phi(b,s) + R\Phi(c,s)]C + [g(b) - g(a)] + [g(c) - g(a)]$$
$$+ N\int_a^b d_s[\Phi(t,s)]\,[g(s) - g(a)] + R\int_a^c d_s[\Phi(t,s)]\,[g(s) - g(a)] = 0.$$

or

$$DC + [g(b) - g(a)] + [g(c) - g(a)] + N\int_a^b d_s[\Phi(t,s)]\,[g(s) - g(a)] + R\int_a^c d_s[\Phi(t,s)]\,[g(s) - g(a)] = 0.$$

Out initial assumption that the homogeneous boundary value problem is incompatable ensures that D is non - singular and hence C is given uniquely as

$$C = -D^{-1}[g(b) - g(a)] - D^{-1}[g(c) - g(a)] - D^{-1}N\int_a^b d_s[\Phi(t,s)][g(s) - g(a)] - D^{-1}R\int_a^c d_s[\Phi(t,s)][g(s) - g(a)].$$

Substituting, this evaluation of C in (2.6), we infer a unique solution of the generalized boundary value problem y(t) of (1.1) satisfying (1.2) is given by

$$y(t) = \int_a^b G(t,s)[g(s) - g(a)]ds + D^{-1}[g(b) - g(a)] + D^{-1}[g(c) - g(a)] + g(t) - g(a),$$

where G is the generalized Green's matrix as given in (2.4) or (2.5).
Thus, the proof of theorem is complete.

Remarks. Let us consider the particular case of the absolutely continuous functions A and g having the expression

$$A(t)=\int_a^t B(s)ds \text{ and } g(t)=\int_a^t h(s)ds$$

where B : [a,c] $\to R^{n\times n}$ and $h:[a,b]\to R^n$.

In the case

$$\int_a^t d[A(s)]\,x(s)=\int_a^b d\left[\int_a^t B(s)ds\right]x(s)=\int_a^t B(s)x(s)ds.$$

and $d[g(t)]=d\left[\int_a^t h(s)ds\right]=h(t).$

Hence the generalized linear differential equation (1.1) is equivalent to the first order linear system

$$y'(t)=B(t)y(t)+h(t) \tag{2.7}$$

and the boundary condition

$$My(a)+Ny(b)+Ry(c)=0 \tag{2.8}$$

If Y is a fundamental matrix of $y'(t)=B(t)y$, then any solution of (2.7) satisfying (2.8) is given by

$$y(t)=\int_a^c G(t,s)h(s)ds,$$

where G is the Green's matrix for the homogeneous boundary value problem and is given by

$$\begin{array}{c}G(t,s)\\ t\in[a,b]\end{array}=\begin{cases}Y(t)Y^{-1}(s)-Y(t)D^{-1}NY(b)Y^{-1}(s)-Y(t)D^{-1}RY(c)Y^{-1}(s) & a\le s\le t\le b\le c\\ -Y(t)D^{-1}NY(b)Y^{-1}(s)-Y(t)D^{-1}RY(c)Y^{-1}(s) & a\le t\le s\le b\le c\\ Y(t)D^{-1}RY(c)Y^{-1}(s), & a\le t\le b\le s\le c\end{cases}$$

and

$$\begin{array}{c}G(t,s)\\ t\in[b,c]\end{array}=\begin{cases}Y(t)Y^{-1}(s)-Y(t)\,D^{-1}\,RY(c)Y^{-1}(s), & a\le b\le s\le t\le c\\ -Y(t)D^{-1}\,RY(c)Y^{-1}(s), & a\le b\le t\le s\le c\\ Y(t)Y^{-1}(s)-Y(t)D^{-1}NY(b)Y^{-1}(s)-Y(t)D^{-1}RY(c)Y^{-1}(s), & a\le s\le b\le t\le c\end{cases}$$

which are precisely same as in [4].

Theorem 2.3. The generalized Green's matrix G defined by (2.4) has the following properties.

(i) G(t,s) as a function of t with s fixed have continuous first derivatives everywhere except at t=s. At the point t=s, G has an upward Jump discontinuity of unit magnitudes.

$$G(s^+,s)-G(s^-,s)=I_n.$$

(ii) G(t,s) is a formal solution of the homogenous three - point boundary value problem and G fails to be the true solution because of the discontinuity at t=s.

(iii) Generalized Green's matrix satisfying properties (i)and (ii) is unique.

Note: For two point boundary value problems associated with (2.7), we refer [1].

3. Kronecker Product of Linear System of Generalized Matrix Differential Equations

In this section, we consider two Linear generalized system of differential equations of the form

$$d_1y = d[A]y + dg, t \in [a,c] \tag{3.1}$$

*a*nd

$$d_2x = d[B]x + df, t \in [a,c] \tag{3.2}$$

where A and B are square matrices of orders $(n \times n)$ and $(m \times m)$continuous matrices and g and $f \in BV^n$ and BV^mrespectively.The Kronecker product (or tensor product) of two matrices A = (a_{ij}) and B = (b_{ij}) are defined as

$(A \otimes B) = (a_{ij}B)fori = 1,2,\dots,m; j = 1,2,\dots,n$ and is of order $(nm \times nm)$. In general :

(I) $(A \otimes B) \neq (B \otimes A)$

(II) $(A \otimes B)(C \otimes D) = (AC \otimes BD)$

(III) $(A \otimes B)^T = (A^T \otimes B^T)$

(IV) $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$

(V) $\frac{d}{dt}(A \otimes B) = \frac{dA}{at} \otimes B + A \otimes \frac{dB}{at}$,

where the matrices involved are of appropriate dimensions to be conformable and invertible. System of equations given by (3.1) and (3.2) can be conveniently recast as

$$d(y \otimes x) = [d(A) \otimes I_m + I_n \otimes d(B)] \tag{3.3}$$

If $\Phi(t,s)$ and $\Psi(t,s)$ be fundamental matrices of the homogenous system (3.1) and (3.2), then we have the following result:

Theorem 3.1: $(\Phi\otimes\Psi)$ is a generalized fundamental matrix of the homogenous system (3.3), if and only if Φ and Ψ are generalized fundamental matrices to

$$d_1y = d[A]y \tag{3.4}$$
$$d_2x = d[B]x, \tag{3.5}$$

respectively.

Proof: First, we observe that the generalized fundamental matrices,$\Phi(t,s)$ and $\Psi(t,s)$ has the following properties

$\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$and so $\Phi^{-1}(t,s) = \Phi(s,t)$ and
$\Phi(t,t) = I_n$and$\Phi(s,s) = I_n$
Similarly, $\Psi(t,s) = \Psi(s,t)$and $\Psi(t,t) = \Psi(s,s) = I_m$.

First, suppose Φ is a generalized fundamental matrix solution of (3.4) and Ψ is a generalized fundamental matrix solution of (3.5). Then consider

$(\Phi\otimes\Psi)^1(t,s) = \Phi^1(t,s)\otimes\Psi(t,s) + \Phi(t,s)\otimes\Psi^1(t,s)$
$= A(t)\,\Phi(t,s)\otimes\Psi(t,s) + \Phi(t,s)\otimes B(t)\Psi(t,s)$
$= [(A(t)\otimes I_m + I_n\otimes B(t)]\ [(\Phi\otimes\Psi)](t,s).$

Hence the claim. Conversely, suppose $(\Phi\otimes\Psi)$ be a generalized fundamental matrix solution of (3.3), ten it can easily be proved as presented in a novel approach by Kasi Viswanadh V Kanuri et.al in [], that Φ is a fundamental matrix solution of (3.4) and Ψ is a fundamental matrix solution of (3.5).

Now, the variation of parameters formula is given in the next theorem.

Theorem 3.2: A particular solution $(\hat{x}\otimes\hat{y})(t)$ of the non homogenous Kronecker product system

$$d(y\otimes x) = [dA\otimes I_m + I_n\otimes B] + [g\otimes I_m + I_n\otimes f] \qquad (3.6)$$

isgiven by

$(\hat{x}\otimes\hat{y})(t) = (\Phi\otimes\Psi)(t,s)(C_1\otimes C_2)(t)$, where
$(C_1\otimes C_2)^1(t) = [\Phi^{-1}(t,s)\otimes\Psi^{-1}(t,s)]\ [g\otimes I_m + I_n\otimes f]$

and hence

$$(C_1\otimes C_2)(t) = \int_a^t [\Phi^{-1}(t,s)\otimes\Psi^{-1}(t,s)]\,[g(s)\otimes I_m + I_n\otimes f(s)]ds$$

Thus

$$(\hat{x}\otimes\hat{y})(t) = (\Phi\otimes\Psi)(t,s)\int_a^t [\Phi(t,s)\otimes\Psi(t,s)]][g(s)\otimes I_m + I_n\otimes f(s)]ds$$

Hence any solution $(x\otimes y)(t)$ of (3.6) is given by

$(x\otimes y)(t) = [\Phi(t,s)\otimes\Psi(t,s)]\ (C_1\otimes C_2) + (\hat{x}\otimes\hat{y})(t)$

Theorem 3.3: Any solution of (3.6) satisfying the boundary conditions
$[M_1\otimes I_n + I_m\otimes N_1][x\otimes y](a) + [M_2\otimes I_m + I_n\otimes N_2][x\otimes y](b) + [M_3\otimes I_m + I_n\otimes N_3][x\otimes y](c)$is given by

$$(x\otimes y(t)) = \int_a^c G(t,s)[g(s)\otimes I_m + I_n\otimes f(s)]ds$$

where$G(t,s)$ is the Green's matrix for the homogenous boundary value problem.

The properties of the Green's matrix are similar to the properties of the Green's matrix presented by Kasi ViswanadhV. Kanuri, et. al [7,10].

REFERENCES:

1. R.H. Cloe, Theory of ordinary differential equations, Application - Century - Gufts- 1968.
2. Halnay A and Moro A. A boundary value problem and its adjoint, Ann. Mat. Pur. Appl. 79, 399 - 412 [1968].
3. Kurzureil J.,Generalized ordinary differential equations and continuous dependence on *parameters, Czechoslovak, Math. J. 82, 418-499 [1957].*
4. Rao, D.R.K.S. Murty K.N. and Murty M.S.N., On three - point boundary value problemscontaining parameters. Bulletin of the institute of Mathematics AcademiaSinica, Vol: 10, No: 3, 265 - 275 [1982].

5. PrepetitaV . Calculus of the fundamental matrix for a generalized differential equations, Ann. Sci. Mat., Anne Bec, Vol 23, No. 1, 87-96 [1996].
6. Kasi Viswanadh V. Kanuri, K. N. Murty, "Three-Point boundary value problems associated with first order matrix difference system-existence and uniqueness via shortest and closest Lattice vector methods", Journal of Nonlinear Sciences and Applications, Volume 12, Issue 11, (2019) 720--727.
7. Kasi Viswanadh V Kanuri, Existence Of Ψ-Bounded Solutions For Fuzzy Dynamical Systems On Time Scales, International Journal of Scientific & Engineering Research, 2020, Vol. 11, No. 5, 613--624.
8. Kasi Viswanadh V. Kanuri, R. Suryanarayana, K. N. Murty, Existence of Ψ-bounded solutions for linear differential systems on time scales, Journal of Mathematics and Computer Science, 20 (2020), no. 1, 1--13.
9. Kasi Viswanadh V Kanuri, Y. Wu, K.N. Murty, Existence of $(\Phi \otimes \Psi)$ bounded solution of linear first order Kronecker product of system of differential equations, International Journal of Science and Engineering Research, 2020, Vol 11, No. 6, P: 156--163.
10. Kanuri, K. V. V., Bhagavathula, S., &Murty, K. (2020). Stability Analysis of Linear Sylvester System of First Order Differential Equations. *International Journal of Engineering and Computer Science*, *9*(11), 25252--25259.
11. K. N. Murty, V. V. S. S. S. Balaram, K.V. K. Viswanadh, "Solution of Kronecker Product Initial Value Problems Associated with First Order Difference System via Tensor-based Hardness of the Shortest Vector Problem", Electronic Modeling, vol. 6, p: 19--33 (2008).
12. Y. Wu, D.L. Nethi, K.N.Murty, Initial Value Problems Associated With First Order Fuzzy Difference System -Existence And Uniqueness, International Journal of Recent Scientific Research. 11 (2020), no. 3, 37846--37848.
13. K.N. Murty, Yan Wu, Viswanadh V. Kanuri, "Metrics that suit for dichotomy, well conditioning of object oriented design", Nonlinear Studies, Vol 18, No: 4, P: 621--637 (2013).