Kronecker product Three point boundary value problems Existence and Uniqueness

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Abstract: In this paper we shall be concerned with the existence and unicity of solutions to first order Kronecker product system satisfying most general boundary conditions at three points. Best least square solutions are presented by Wi chobesky decomposition and QR-algorithm in the non-invertible case of the characteristic matrix.

Keywords: Kronecker product of Matrices, Linear systems Chobesky decomposition, QR algorithm, best Least square solutions.
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1. Introduction

This paper presents a criteria for the existence and unicity of solutions to three-point boundary value problems associated with first order Kronecker product system on the interval $a < b < c$. Kronecker product boundary value problems is an interesting area of current research and many results came in the recent years. The first result in this direction is due to Don Fausett and K.N.Murthy [2] and later Kasi Viswanadh, Anand,Rompicharla,Yan Wu,Vellanki Lakshmi,Sailaja et.al, established on $(\Phi\otimes\Psi)$ bounded solutions and $(\Phi^{\alpha}\otimes\Psi^{\alpha})$ bounded solution to Kronecker product linear first order systems [9]. Motivated by the results established by Kasi viswanadh et.al[1,7,9,10]. In 2012 V S Putcha et.al [18] developed variation of parameters formula for fuzzy matrix Discrete systems. V S Putcha et.al [19] established existence of Ψ- Bounded Solutions for First Order Matrix Difference System on Z in 2019. In 2009 P.V.S.Anand [17] established some results on Controllability and Observability of continuous and discrete Matrix Lyapunov Systems. In 2019 Charyulu L. N. Rompicharla et. al [7] established Controllability and observability of fuzzy matrix discrete dynamical systems. Recently in 2020 Charyulu L. N. Rompicharla et. al [6] established the existence of Ψ bounded and $(\Phi \otimes \Psi)$ bounded solutions of linear first order Kronecker product difference solution

We established existence and uniqueness criteria to three-point boundary value problems. When the characteristic matrix is either rectangular or singular uniqueness of solutions in the best least square solutions are presented using Chobesky decomposition and modified QR algorithm.

We consider two first order non homogeneous linear system of equations is of the form

$$y^{l} = A(t)y + f(t) \tag{1.1}$$

and

$$x^{l} = B(t)y + g(t) \tag{1.2}$$

and the homogeneous system of (1.1)and (1.2) be in

$$y^{l} = A(t)y \tag{1.3}$$

and $$x^l = B(t)x \tag{1.4}$$

where $A(t)$ and $B(t)$ are $(m \times m)$ and $(n \times n)$ continous matices respectively and y and z are column matrices of order m and n respectively.(1.3) and (1.4) can be embedded in a single frame work by using Kronecker product of matrices as

$$(y \otimes x)^l = [(A(t) \otimes I_n) + (I_m \otimes B(t))][x \otimes y] \tag{1.5}$$

And the non homogeneous equations (1.1)and (1.2) can be written as

$$(y \otimes x)^l = [(A(t) \otimes I_n) + (I_m \otimes B(t))][y \otimes x] + [f \otimes I_n + I_m \otimes g](t) \tag{1.6}$$

Where I_n and I_m are unit matrices of orders n and m respectively. Let $\Phi(t)$ be a fundamental matrix of (1.3) and let $\Psi(t)$ be a fundamental matrix of (1.4). Then $[\Phi(t) \otimes \Psi(t)]$ is a fundamental matrix of (1.5). For the proof of this, we refer Kasi Viswanadh, et.al[ ]. This paper is organized as follows: Section 2 , presents variation of parameters formula for (1.6) and also presented some sailent features of Kronecker product of matrices. Section 3 presents a criteria for the existence and uniqueness of three-point boundary value problems. Finally, we establish best least square solutions of three-point boundary value problems by using QR-algorithm and chobesky decomposition.

2. Preliminaries

In this section, we shall be concerned with the Kronecker product boundary value problem and establish variation of parameters formula for (1.6). We now present the properties of the Kronecker product of matrices.

Definition 2.1: If $A \in R^{m \times m}$ and $A \in R^{n \times n}$ the their Kronecker product of A and B denoted by $A \otimes B$ is defined as $(A \otimes B) = (a_{ij}B)$ for all $i, j = 1,2, \dots, m$.

The Kronecker product of matrices defined above has the following properties.

(i) $(A \otimes B)$ is not same as $(B \otimes A)$
(ii) $(A \otimes B)^T = (A^T \otimes B^T)$
(iii) $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$
(iv) $A \otimes (B + C) = A \otimes B + A \otimes C$
(v) $\frac{d}{dt}(A \otimes B) = \frac{dA}{at} \otimes B + A \otimes \frac{dB}{at}$
(vi) $(A \otimes B)(C \otimes D) = (AC \otimes BD)$

where the Matrices involved are of appropriate dimensions to be conformable for multiplication and invertible.

Let $\Phi(t)$ be a fundamental matrix of (1.3), then any solution of (1.3) is of the form $\Phi(t)C_1$, where C_1 is a constant m vector and let $\Psi(t)$ be a fundamental matrix of (1.4), then any solution of (1.4) is of the form $\Psi(t)C_2$, where C_2 is a constant n vector. Any solution of the Kronecker product system (1.5) is of the form $(\Phi(t) \otimes \Psi(t))(C_1 \otimes C_2)$. We now turn our attention to the general solution of (1.6). Let $(y(t) \otimes x(t))$ be any solution of (1.6) and $(\overline{y}(t) \otimes \overline{x}(t))$ be a particular solution of (1.6). Then it can easily verified that $(y(t) \otimes x(t)) - (\overline{y}(t) \otimes \overline{x}(t))$ is a solution of (1.5). Hence

$\big(y(t)\otimes x(t)\big) - (\overline{y}(t)\otimes\overline{x}(t)) = \big(\Phi(t)\otimes\Psi(t)\big)(C_1\otimes C_2).$

or

$\big(y(t)\otimes x(t)\big) = (\overline{y}(t)\otimes\overline{x}(t)) + \big(\Phi(t)\otimes\Psi(t)\big)(C_1\otimes C_2).$

We seek a particular solution of $(\overline{y}(t)\otimes\overline{x}(t))$ in the form

$\big(\overline{y}(t)\otimes\overline{x}(t)\big) = \big(\Phi(t)\otimes\Psi(t)\big)(C_1\otimes C_2)(t).$

Substituting this form in (1.6), we get

$$\Big[\big(\Phi^l(t)\otimes\Psi(t)\big) + \big(\Phi(t)\otimes\Psi^l(t)\big)\Big](C_1\otimes C_2)(t) + \big(\Phi(t)\otimes\Psi(t)\big)[(C_1\otimes C_2)]^l(t)$$
$$= \big[(A(t)\otimes I_n) + \big(I_m\otimes B(t)\big)\big](C_1\otimes C_2)(t) + [f\otimes I_n + I_m\otimes g(t)]$$

$$\big(A(t)\Phi(t)\otimes\Psi(t)\big) + \big(\Phi(t)\otimes B(t)\Psi(t)\big)(C_1\otimes C_2)(t) = [A(t)\otimes I_n + I_m\otimes B(t)](C_1\otimes C_2)(t) + [f\otimes I_n + I_m\otimes g(t)]$$

$$[A(t)\otimes I_n + I_m\otimes B(t)](C_1\otimes C_2)(t) + \big(\Phi(t)\otimes\Psi(t)\big)[(C_1\otimes C_2)]^l(t)$$
$$= [A(t)\otimes I_n + I_m\otimes B(t)](C_1\otimes C_2) + [f\otimes I_n + I_m\otimes g(t)]$$

Hence $(\Phi\otimes\Psi)(t)[(C_1\otimes C_2)]^l(t) = [f\otimes I_n + I_m\otimes g(t)].$

Therefore

$(C_1\otimes C_2)^l(t) = (\Phi^{-1}\otimes\Psi^{-1})(t)[f\otimes I_n + I_m\otimes g(t)]$

Or

$(C_1\otimes C_2)(t) = \displaystyle\int_a^t [\Phi^{-1}(s)\otimes\Psi^{-1}(s)]\,[f(s)\otimes I_n + I_m\otimes g(s)]ds$

Substituting this general form of $(C_1\otimes C_2)$ in the particular solution, we get

$\big(\overline{y}(t)\otimes\overline{x}(t)\big) = [\Phi(t)\otimes\Psi(t)]\int_a^t[\Phi^{-1}(s)\otimes\Psi^{-1}(s)]\,[f(s)\otimes I_n + I_m\otimes g(s)]ds$ (2.1)

3. Existence and Uniqueness: In this section we established existence and uniqueness criteria for the three point boundary value problems.

$y^l = A(t)y + f(t), \quad a \le t \le c$ (3.1)

$M_1 y(a) + N_1 y(b) + R_1 y(c) = \alpha$

and $x^l = B(t)y + g(t), \quad a \le t \le c$

$M_2 x(a) + N_2 x(b) + R_2 x(c) = \beta$

By embedded in it into a single kronecker product three point boundary value problem as

$$(y\otimes x)^{l}(t) = [A(t)\otimes I_n + I_m\otimes B(t)](y\otimes x)(t) + [f\otimes I_n + I_m\otimes g(t)] \qquad a \le t \le c$$

$$M(y\otimes x)(a) + N(y\otimes x)(b) + R(y\otimes x)(c) = (\alpha \times \beta)$$

where $M = (M_1\otimes I_n + I_m\otimes M_2), N = (N_1\otimes I_n + I_m\otimes N_2)$ and $R = (R_1\otimes I_n + I_m\otimes R_2)$ are square matrices of order $(mn \times mn)$. We establish existence and uniqueness criteria in terms of an integral equation involving Green's matrix and the properties of the Green's matrix will be presented in theorem 3.1. We assume that the homogeneous boundary value problem with $f(t) = g(t) = 0$ $w(\alpha \times \beta) = 0$ in (3.2) has only the trivial solution. This assumption clearly ensures that the Non Homogeneous boundary value problem has a unique solution. In the next section, we relax this assumption and establish best least square solution of (3.2) by page 5 line 5 chobesky decomposition and modified QR algorithm. Substituting the general form of solution given by (2.1) in the boundary condition matrix, we get

$$\begin{aligned}&[M(\Phi\otimes\Psi)(a) + N(\Phi\otimes\Psi)(b) + R(\Phi\otimes\Psi)(c)](\mathrm{C}_1\otimes \mathrm{C}_2)\\ &\quad + N[\Phi(\mathrm{b})\otimes\Psi(\mathrm{b})]\int_a^b [\Phi^{-1}(s)\otimes\Psi^{-1}(s)]\,[f\otimes I_n + I_m\otimes g(s)]ds\\ &\quad + R[\Phi(\mathrm{c})\otimes\Psi(\mathrm{c})]\int_a^c [\Phi^{-1}(s)\otimes\Psi^{-1}(s)]\,[f\otimes I_n + I_m\otimes g(s)]ds = (\alpha\otimes\beta)\end{aligned}$$

Let D be the characterstic matrix defined by

$$D = M(\Phi(\mathrm{a})\otimes\Psi(\mathrm{a})) + N(\Phi(\mathrm{b})\otimes\Psi(\mathrm{b})) + R(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c})).$$

Then

$$\begin{aligned}(\mathrm{C}_1\otimes \mathrm{C}_2) &= -D^{-1}[N\Phi(\mathrm{b})\otimes\Psi(\mathrm{b})]\int_a^b [\Phi^{-1}(s)\otimes\Psi^{-1}(s)]\,[f\otimes I_n + I_m\otimes g(s)]ds\\ &\quad + R[\Phi(\mathrm{c})\otimes\Psi(\mathrm{c})]\int_a^c [\Phi^{-1}(s)\otimes\Psi^{-1}(s)]\,[f\otimes I_n + I_m\otimes g(s)]ds + D^{-1}(\alpha\otimes\beta)\end{aligned}$$

Substituting this general form of $(\mathrm{C}_1\otimes \mathrm{C}_2)$ in (2.1) we get

$$(\mathrm{y}(\mathrm{t})\otimes \mathrm{x}(\mathrm{t})) = \int_a^c G(t,s)\,(f\otimes I_n + I_m\otimes g(s))ds + D^{-1}(\alpha\otimes\beta).$$

Where $G(t,s)$ is a Green's matrix for the homogeneous boundary value problem and is given by

$$G(t,s)_{t\in[a,b]} = \begin{cases} \big(\Phi(\mathrm{t})\otimes\Psi(\mathrm{t})\big)D^{-1}M\big(\Phi(\mathrm{t})\otimes\Psi(\mathrm{t})\big)\big(\Phi^{-1}(s)\otimes\Psi^{-1}(s)\big) a < s < t \le b \le c\\ -\big(\Phi(\mathrm{t})\otimes\Psi(\mathrm{t})\big)D^{-1}[N\big(\Phi(\mathrm{b})\otimes\Psi(\mathrm{b})\big) + R\big(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c})\big)[\Phi^{-1}(s)\otimes\Psi^{-1}(s)] a \le t < s < b < c\\ -\big(\Phi(\mathrm{t})\otimes\Psi(\mathrm{t})\big)D^{-1}R\big(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c})\big)\big(\Phi^{-1}(s)\otimes\Psi^{-1}(s)\big)\; a < t < b < s \le c\end{cases}$$

and

$$G(t,s)_{t\in[b,c]} = \begin{cases} (\Phi(\mathrm{t})\otimes\Psi(\mathrm{t}))D^{-1}M(\Phi(\mathrm{a})\otimes\Psi(\mathrm{a}))(\Phi^{-1}(s)\otimes\Psi^{-1}(s)) a < s < t \le b \le c \\ -(\Phi(\mathrm{t})\otimes\Psi(\mathrm{t}))D^{-1}[N(\Phi(\mathrm{b})\otimes\Psi(\mathrm{b})) + R(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c}))[\Phi^{-1}(s)\otimes\Psi^{-1}(s)] a \le t < s < b < c \\ -(\Phi(\mathrm{t})\otimes\Psi(\mathrm{t}))D^{-1}R(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c}))(\Phi^{-1}(s)\otimes\Psi^{-1}(s))\ a < t < b < s \le c \end{cases}$$

Theorem 3.1: Green's matrix defined above when consider as a function of t for a fixed s has the following properties

1. G is continuous and possesses continuous first order partial derivatives at all points except at $t = s$ and at the point $t = s$, G has an upward jump discontinuity of limit magnitude i.e,
   $G(s^+, s) - G(s^-, s) = I_{mn}$
2. G is a formal solution of the homogeneous boundary value problem and it fails to be a because of the discontinuity at $t = s$.
3. G is unique with properties 1 and 2.

Proof: We first prove the above three properties in the first case, namely when $t \in [a, b]$. Consider

$G(s^+, s) - G(s^-, s) = (\Phi(\mathrm{s})\otimes\Psi(\mathrm{s}))D^{-1}M(\Phi(\mathrm{a})\otimes\Psi(\mathrm{a}))(\Phi^{-1}(s)\otimes\Psi^{-1}(s)) + (\Phi(\mathrm{s})\otimes\Psi(\mathrm{s}))D^{-1}N(\Phi(\mathrm{b})\otimes\Psi(\mathrm{b})) + R(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c}))(\Phi^{-1}(s)\otimes\Psi^{-1}(s))$

$= (\Phi(\mathrm{s})\otimes\Psi(\mathrm{s}))D^{-1}[M(\Phi(\mathrm{a})\otimes\Psi(\mathrm{a})) + N(\Phi(\mathrm{b})\otimes\Psi(\mathrm{b})) + R(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c}))][\Phi^{-1}(s)\otimes\Psi^{-1}(s)]$

$= (\Phi(\mathrm{s})\otimes\Psi(\mathrm{s}))D^{-1}D[\Phi^{-1}(s)\otimes\Psi^{-1}(s)]$

$= I_{mn.}$

Thus property 1 is verified. The representation $G(t, s)$ in each interval clearly shows, it is a solution of the kronecker product homogeneous system. Now, to show that G satisfies homogeneous boundary conditions, consider

$$\begin{aligned} M\,G(a,s) + N\,G(b,s) + R\,G(c,s) \\ &= [-M(\Phi(\mathrm{a})\otimes\Psi(\mathrm{a})) + N(\Phi(\mathrm{b})\otimes\Psi(\mathrm{b})) + R(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c}))]D^{-1}N(\Phi(\mathrm{b})\otimes\Psi(\mathrm{b}))(\Phi(\mathrm{s})\otimes\Psi(\mathrm{s})) \\ &- [M\,\Phi(\mathrm{a})\otimes\Psi(\mathrm{a}) + N[\Phi(\mathrm{b})\otimes\Psi(\mathrm{b}) + \mathrm{R}(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c}))(\Phi(\mathrm{s})\otimes\Psi(\mathrm{s}))] \\ &+ N[\Phi(\mathrm{b})\otimes\Psi(\mathrm{b})\Phi^{-1}(s)\otimes\Psi^{-1}(s)] + R(\Phi(\mathrm{c})\otimes\Psi(\mathrm{c}))(\Phi^{-1}(s)\otimes\Psi^{-1}(s)) = 0. \end{aligned}$$

Now, to prove uniqueness, Let $G(t, s)$ be another Green's matrix satisfying the properties 1 and 2. Write

$H(t, s) = G(t, s) - G_1(t, s).$

Then H is continuous and possess continuous first order derivatives at all points except at $t = s$ and

$H(s^+, s) - H(s^-, s) = 0.$

It follows that H has a removable discontinuity and so by redefine H appropriately, we can make H continuous and h satisfies the homogeneous linear system and boundary conditions. By our initial assumption that the homogeneous boundary value problem has only the initial solution ensures $H(t,s) \equiv 0$ and hence $G(t,s) \equiv G_1(t,s)$.

Thus uniqueness is established.

4. Best Least square solutions.

In this section, we develop methods to find the best Least square solutions of the kronecker product matrix system of the form

$(A \otimes B)(y \otimes x) = (\alpha \otimes \beta)$ (4.1)

Where A is an $(m \times n)$ matrix and B is a $(p \times q)$ matrix. Make matter simple, we put $A \otimes B = D$ where D is an $(mp \times nq)$ matrix and $Z = (y \otimes x)$ is an $(mp \times 1)$ vector and $(\alpha \times \beta) = b$ also a given $(mp \times 1)$ column matrix. With the notion (4.1) takes the form

$DZ = b$ (4.2)

Further put $mp = k$ and $nq = l$ then D is a $(k \times l)$ sparse matrix. Assume that columns of D are linearly independent$(k > l)$. In this case the unique solution of the system of equations is given by $Z = (D^T D)^{-1}\, b$.

If the rows of D are linearly independent , then $Z = D^T y$ gives the

$DD^T y = b$ and $y = (DD^T)^{-1}\, b$

Or $Z = D^T y = D^T (DD^T)^{-1}\, b$

Definition 4.1: Let $S \subset R^{pq}$. The orthogonal complement of S denoted by $S^{\perp}$ identified as the set of all vectors $\alpha \in R^{pq}$ that are orthogonal to S.

One important property of orthogonal complement is that $R^{pq} = V \oplus V^T$ where $\oplus$ is the direct sum which means any vector $\alpha \in R^{pq}$ can uniquely be written as $Z = b + 0$ where $p \in V$ and $0 \in V^{+}$.

Theorem 4.1: Let D be an $(mn \times pq)$ matrix with $mn > pq$ and assume that columns of D are linearly independent then the system of equations are $DZ = b$ (4.1)

has a least square solution $\bar{Z}$ if, and only if it is a solution of the augumented matrix system

$$D^T DZ = D^T b \quad (4.2)$$

Proof: Let $Z \in R^{pq}$. Then DZ is an arbitrary vector in the column space of D, which we write as R(D).Let $r(Z) = b - DZ$

is minimum if DZ is the orthogonal projection of α onto $R(D)$. Since $R(D)^{\perp} = null(D^T)$. $\bar{Z}$ is a least square solution if, and only if $D^T r(b) = D^T (b - D\bar{Z}) = 0$ which is equal to the system of normal equations of the form $D^T DZ = D^T b$

For this solution to be unique, the matrix D must have full column rank of D.

Theorem 4.2: Consider a system of linear equations DZ=b and associated normal system of equations $D^T DZ = D^T b$.

Then the following are equivalent.

1. The least square problem has a unique solution.
2. The linear system DZ=0 has only the trivial solution.
3. The columns of D are linearly independent.

Proof: Proof is elementary and hence omitted.

Theorem 4.3: If S is a vector space of finite dimension then all norms on S are almost equivalent i.e.

$$C\|Z\|_n \leq \|Z\|_m \leq C\|Z\|_n$$

Where $0 < C \leq \mathcal{C}$ and for $Z \in S$.

Proof: It is sufficient to show that the result in the for any two norms on the unit sphere. For any given vector $Z \in R^n$ we can write $Z = \gamma Z_0$ where $\gamma = \|Z\|_n$ and Z_0 is a vector on the unit sphere. First, we note that any norm is a continous function. For $Z_0 \in R^n$ for every $Z \in R^n$, the triangle inequality gives

$\|Z\| - \|Z_0\| \leq \|Z - Z_0\|$

Thus $\|Z\| - \|Z_0\| < \varepsilon$ whenever $\|x - Z_0\| < \varepsilon$. For simplicity, we work the case for n=2.

Let e_i be the canonical case in R^n, then write $Z = \bar{Z} Z_i e_i$. So that

$\|Z\|_m \leq \sum |Z_i| \|e_2\|_m \leq \sqrt{\sum Z_i^2} \sqrt{\bar{Z} \|e_i\|^2} = \mathcal{C} \|Z\|_2$

By using the continuity on the unit sphere, it can be shown that as $Z \rightarrow \|Z\|_m \geq C \|Z\|_2$.

With the equivalence of norm, we choose the 2-norm in our future discussion to find he least square solution of the boundary value problem.

Definition 4.2: Suppose that V is an inner product space the norm of the length of the vector $u \in V$ is defined as $\|u\| = (u, u)^{\frac{1}{2}}$

For the sake of simplicity, we take $mn = k$ and $pq = l$ so that D is a $(k \times l)$ matrix and Z is a column vector and b is also a given l vector.

Chobesky Factorization: If D has a full column rank then the following are true for $D^T D$:

1. $D^T D$ is symmetric i.e $(D^T D)^T = D^T (D^T)^T = D^T D$
2. $D^T D$ is positive definite i.e.
   $(Z^T D^T D Z) = (DZ)^T DZ = \|DZ\|^2 \geq 0$ if $Z \neq 0$

Thus if D is a $(k \times l)$ matrix with columns of D are L.I then $D^T D$ is a positive definite symmetric matrix.

In this case it is favorable to use Chobesky Factorization, which decomposes$D^T D = L L^T$, where L is an $(n \times n)$ lower triangular matrix.

Flop: Triangular matrices are used extensively in numerical algorithms such as Chobesky or QR Factorization because triangular system are one of the simplest to solve system of equations. We introduce the Flop count for triangular system. A Flop is a floating point operation $(+, -, *, /)$, in any $(l \times l)$ lower triangular matrix $LZ = b$

each Z_k in Z is obtained by operating $Z_k = b_k - \sum_{j=1}^{k-1} l_{kj} Z_j$

which only requires $(k-1)$ multiplications on $(k-1)$ additions. Thus Z requires $l^2 - l$ flops to complete. Since l is very large for sparce matrices, we ignore lower order terms. We say that an $(l \times l)$ forward substitution costs $\sim l^2$ Flops.

Algorithm: Computing the Chobesky Factorization: For a matrix D define

$D_i : i^l, j : j^l$ as $\left(i^l - i + 1\right) * \left(j^l - j + 1\right)$ sub matrix of D with upper left corner D_{ij} and lower right corner ${D_{ij}}^l$

Algorithm 4.1:

$R = D$

For $K = 1$ to m

For $j = k + 1$ to m

$R_{j,j:m} = R_{j,j:m} - R_{k,j:m} \quad {}^{R_{kl}}/_{R_{lk}}$

$R_{k,k:m} = R_{k,k:m} / SQRT(R_{kk})$

The last but one line dominates the operation count, so its flop count is

$\sum_{k=1}^{m} \sum_{j=k+1}^{m} 2(m-j) \sim 2 \sum_{k=1}^{m} \sum_{j=1}^{k} j \sim \sum_{k=1}^{m} k^2 \sim m^3/3.$

Thus, we have the following flop count

1. Calculate $C = D^T D$ $(\simeq kl^2\ Flops)$
2. Chobesky Factorization $C = LL^T$ $(L^3/3\ Flops)$
3. Calculate $d = D^T b$ $(2kl\ Flops)$
4. Solve $Lx = d$ by forward substitution $(l^2\ flops)$
5. Solve $L^T Z = b$ by backward substitution $(l^2\ flops)$

This gives the cost for large k, l: $kl^2 + \frac{1}{3}l^3\ \ flops$.

Result: Let D be a $(k \times l)$ given matrix with rank of $D = b \leq min\{k, l\}$. Then there exists a Factorization of the form DP=QR with the following properties:

1. P is a $(l \times l)$ permutation matrix with the first D columns of the DP form a basis of
$$Im(D) = \{DZ \in R^k / Z \in R^l\}$$
2. Q is a $(k \times b)$ matrix with orthogonal columns and R is $(p \times l)$ upper trapezoidal matrix of the form $R = [R_1\ R_2]$ where $R_1$is non singular $(p \times p)$ upper triangular matrix and R_2 is $(p \times p)$ matrix.

Now, we consider $D = (A \otimes B)(x \otimes y) = (\alpha \times \beta)$ (4.3)

Where A is $(m \times n)$ and B is $(p \times q)$ rectangular matrices. Suppose that A and B are QR-decomposed as $A = Q_1 R_1$ and $B = Q_2 R_2$, where Q_1 is $(m \times m)$ with ortho normal column, R_1 is $(m \times n)$ upper trapezoidal matrix and Q_2 is $(p \times p)$ matrix with ortho normal columns and R_2 is $(p \times q)$ upper trapezoidal matrix. Then
$$(A \otimes B) = (Q_1 R_1) \otimes (Q_2 R_2) = (Q_1 \otimes Q_2)(R_1 \otimes R_2)$$

Theorem 4.4: If $A = Q_1 R_1$ and $B = Q_2 R_2$, then $(A \otimes B)$ has a has a permitted QR-factorization with $(Q_1 \otimes Q_2)$ is orthogonal and $(A \otimes B) = (Q_1 R_1 \otimes Q_2 R_2) = (Q_1 \otimes Q_2)(R_1 \otimes R_2)$

Proof: Consider $(Q_1 \otimes Q_2)^T (Q_1 \otimes Q_2) = \left({Q_1}^T \otimes {Q_2}^T\right)(Q_1 \otimes Q_2)$
$$= \left({Q_1}^T Q_1 \otimes {Q_2}^T Q_2\right)$$
$$= \left(I_m \otimes I_p\right)$$
$$= I_{mp}.$$

This implies that $(Q_1 \otimes Q_2)$ is orthogonal and $Z = \begin{pmatrix} Z \\ 0 \end{pmatrix} = (R_1 \otimes R_2)$

Where C is $(nq \times nq)$ matrix and O is an $(mp - np) \times nq$ matrix and Z is permutation matrix.

If we write $R_1 = \begin{bmatrix} r_{12}^{(1)} & r_{12}^{(1)} & \cdots & r_{1l}^{(1)} \\ 0 & r_{22}^{(1)} & \cdots & r_{2l}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & r_{ll}^{(1)} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} R^{(1)} \\ 0 \end{bmatrix}$

Definition 4.3: Let $S \subset R^{kl}$. The orthogonal complement of S denoted by $S^{\perp}$ is defined as the set of all vectors $Z \in R^{kl}$ that are orthogonal to S. One important property of orthogonal complement is that $R^{kl} = V \oplus V^T$. Where $\oplus$ is the direct sum which means any vector $Z \in R^{kl}$ can be written uniquely as Z=p+0 where $p \in V$ and $0 \in V^+$.

Theorem 4.5: Let D be an $(mn \times pq)$ matrix which means L.T and $\alpha \in R$. And $\bar{Z}$ is a least square solution of the system DZ=b, where $D = (A \otimes B)$ and $(\alpha \times \beta) = b$ if,and only if it is a solution of the augumented linear system $D^T DZ = D^T b$

Example 4.1: Let $D = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$ rank(D)=2.

With five digits accuracy after computing QR-Algorithm, we get $Q = \begin{bmatrix} 0.70710 & 0.70711 \\ 0.70711 & 0.70710 \end{bmatrix}$

Clearly Q is not orthogonal. If such a Q is used to find a least square solution, we find that the system would be highly sensitive to permutations and hence we need to apply minimum norm solution by using singular value decomposition.

Example 4.2: Consider the system of equations $DZ = b$
where $D = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ 1 & 3 & -3 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 15 \\ 6 \\ 22.5 \end{bmatrix}$
Here A is (3x1) matrix and B=(1x4) matrix and D is a (3x4) rectangular matrix with columns linearly independent. The minimum least square solution by Wie Chobesky decomposition is given.
$\bar{Z} = [-0.2110092, -0.633028, 0.963303]^T$
The best Least square solution of the Kronecker product system with QR-Algorithm is given by $Z = [-0.210008, -0.633128, -0.973403]^T$.

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