Existence of Ψ^{α} Bounded Solutions of Linear First Order Fuzzy Lyapunov Systems- A New Approach

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Keywords: Fuzzy sets and systems, Lyapunov systems, Stability Analysis and Fundamental fuzzy matrix solutions.

AMS (MoS) classifications: 34A07, 26A39, 46G05

1. Introduction

This paper presents a criteria for the existence of Ψ^{α} bounded solution of the linear first order Fuzzy Lyapunov system of the form

$$(T^{\alpha})'(t) = A(t)T^{\alpha}(t) + T^{\alpha}(t)A^{*}(t) + F(t), \tag{1.1}$$

where $A(t)$, $A^{*}(t)$ are square matrices of order (nxn) and F(t) is a square matrix of order n and $T^{\alpha}(t)$ is an (nxn) continuous matrix and $\alpha \in [0,1]$. The homogeneous system corresponding to (1.1) is

$$(T^{\alpha})'(t) = A(t)T^{\alpha}(t) + T^{\alpha}(t)A^{*}(t). \tag{1.2}$$

Lyapunov first order system plays an important role in the theory of differential equations and has significant impact on control engineering problems. Recently, much attention has been paid by many authors to study qualitative properties of linear systems and Ψ-bounded solutions of linear first order systems. Among them the results established by Kasi Viswanadh, Yan Wu, K. N. Murty, Rompicherla, Anand, Divya and Pagilla [2-10, 14, 15] need a special mention. Kasi Viswanadh in fact initiated [3] the idea of fuzzy first order systems and his co-authors Narayana S. Ravada, Murty K.N extablished Ψ-bounded solutions Linear systems on time scale dynamical system needs a special mention [4], as it unifies both continuous and discrete systems in a single framework. In fact the existence of Ψ^{α} bounded solutions was first initiated by Kasi Viswanadh V. Kanuri [3] and (Φ⊗Ψ) –bounded solutions by Rompicherla, Anand [14] on Kronecker product linear system of differential equations. Motivated by these ideas, we investigate $(\Psi^{\alpha}, \Psi^{\alpha^{*}})$ bounded solution of Lyapunov linear system of first order fuzzy differential equation (1.1). This paper is organized as follows: section 2, presents basic results on fuzzy sets and systems and develop variation of parameters formula for (1.1). We study existence of $(\Psi^{\alpha}, \Psi^{\alpha^{*}})$ bounded solutions in section 3. We also establish stability criteria for the Lyapunov homogeneous fuzzy system and deduce existing results as particular case.

Recently much attention has been paid by many authors on the existence of (Φ,Ψ)-bounded solutions of linear first order systems [14,16]. The use of Ψ^{α} bounded solutions for fuzzy first order systems is due to Kasi Viswanadh [3] and based on this Kronecker product systems were studied by Rompicherla, Anand et.al [14]. In fact the results established in K.V.K. Viswanadh [4] are very useful to understand α–level sets and have significant impact on characterization of soils and thereby increasing production on food grains to farmers [15].

2. Preliminaries

In this section, we present basic notions on fuzzy sets and systems and define α –level sets for understanding the use of fuzzy sets. Most of the results presented in this section form a clear understanding of our main results. For, let X be a non-empty set. A Fuzzy set A ∈ X is characterized by its membership function A: X→ [0,1] is used to represent intermediate degree of membership. The mapping A is also called the membership function of fuzzy set A. the integrals of fuzzy-number-valued functions is obtained as a natural generalization of set valued functions and have been discussed by Puri and Ralescu [17], Kaleva [1] and other co-authors in [16].

Let T be a closed interval on R i.e, $T = [a, b]$ $(a, b \in R)$

Let $|T|$ denotes the length of T. If there exists $T_i \subset T$, $\xi_i \in T_i$ $(i = 1,2, \dots m)$such that

$\bigcup_{i=1}^{m} T_i = T$ (where $T_1, T_2, \dots, T_m$) are non-overlapping subintervals of T, then a collection

$\{(\xi_1, T_1), (\xi_2, T_2), \dots, (\xi_m, T_m)\}$ is called a division of T and we write

$\Pi = \{(\xi_1, T_1), (\xi_2, T_2), \dots, (\xi_m, T_m)\}$

For clarity, we write $\Pi = \{\xi, [u, v]\}$ where $[u, v]$ denotes a typical interval of Π and ξ is an associated point of $[u, v]$.

Throughout this paper, we $P_k(R^m)$ to denote the family of all non-empty compact convex subsets of R^m.

For $A, B \in P_k(R^m)$, $k \in R$, the addition and scalar multiplication are defined in the usual way as

$A + B = \{x + y / x \in A, y \in B\}, \alpha A = \{\alpha x / x \in A\}$

In addition, for $A, B \in P_k(R^m)$, the Haussdorff metric between them is defined as

$d(A, B) = \max\{sup_{a \in A}\, inf_{b \in B}(\| -a - b\|, sup_{b \in B}\, inf_{a \in A}\|b - a\|\}$.

Defintion 2.1: For A $\in P_k(R^m)$, $x \in S^{m-1}$, the support function of A is defined as

$\sigma(x, A) = sup_{y \in A}\,(y, x)$,

where S^{m-1}denotes the unit sphere of R^m, (.,.) is the inner product of R^m.

It is clear now that for A, B $\in P_k(R^m)$, $x \in S^{m-1}$, we have

(i) $\sigma(x, A) = k\, \sigma(x, A)(k \geq 0)$
(ii) $\sigma(x, A + B) = \sigma(x, A) + \sigma(x, B)$.

Theorem 2.1: Let $A, B \in P_k(R^n)$. Then $d(A, B) = sup_{x \in S^{n-1}}|\sigma(x, A) - \sigma(x, B)|$.

Proof: For the proof of the theorem, we refer P. Diamand and P. Kloeden [18].

Definition 2.2**:** Let $E^n = \{u / u: R^n \to [0,1]$. For any $u \in E^n$. U is said to be ab n-dimensional fuzzy number if it satisfies the following conditions.

(i) u is normal i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$
(ii) u is fuzzy convex i.e., $u[tx + (1 - y)] \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $t \in [0,1]$
(iii) u is upper semi continuous
(iv) $[u]^\alpha = \{x \in R^n: u(x) \geq 0\}$ is compact.

We define $D: E^n X E^n \to [0, \infty)$ by the equation

$D(u, v) = sup_{r \in [0,1]} d[[u]^r, [v]^r], u, v \in E^n$,

then the metric space$\{E^n, D\}$ has a linear structure, it can be imbedded isomorphically as a convex cone with vertex θ in the Banach space of functions.

$u^*: I \times S^{n-1} \to R$ where S^{n-1} is the unit sphere in R^n with an embedding function $u^* = j(u)$ defined as

$u^* = (r, x) = sup_{\alpha \in [u]} < \alpha, x >$ for all $< r, x > \in I \times S^{n-1}$.

Definition 2.3: A fuzzy number in parametric form is represented by (u_α^-, u_α^+), where

$u_\alpha^- = \min[u]^\alpha$ and $u_\alpha^+ = \max[u]^\alpha$, $0 \leq \alpha \leq 1$, and has the following properties:

(i) u_α^- is a bounded left continuous monotonic increasing
(ii) u_α^+ is a left continuous monotonic decreasing function of α over [0,1].
(iii) $u_\alpha^- \leq u_\alpha^+$ for $\alpha \in [0,1]$.

If $f: R^n \to R^n$, is a function then according to Ladeh's principle, we can extend $f: E^n X E^n$ as

$f(u,v)(z) = sup_{z=f(x,y)} \min\{u(x), u(y)\}$ and

Further,

$$[f(u,v)]^\alpha = f([u]^\alpha, [v]^\alpha).$$

For all $u, v \in E^n$ and $\lambda \in R$ and $\alpha \in [0,1]$, the sum $(u+v)$ and the product λu are defined as

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha$$

$$[\lambda u]^\alpha = \lambda [u]^\alpha.$$

Definition 2.4: we define $D: E^n X E^n \to R_+ u\{0\}$ by

$$D(u,v) = \sup d_H([u]^\alpha, [v]^\alpha),$$

And D_H is the Haussdorff metric defined $P_k(R^n)$.

We are now in a position to develop the general solution of the Lyapunov fuzzy homogeneous system (1.2) and then the variation of parameters formula for (1.1).

Theorem 2.2: Any solution of the linear Lyapunov system (1.2) is of the form $Y^\alpha C Y^{*\alpha}$ where Y^α is a fundamental matrix solution of $(T^\alpha)' = A(t)T^\alpha$ for any fixed $\alpha \in [0,1]$.

Proof: Let $\alpha \in [0,1]$ be fixed and let $Y^\alpha(t)$ be a fundamental matrix solution of

$(T^\alpha)' = A(t)T^\alpha$. Then let $\mathcal{K}^\alpha$ be a matrix defined by $\mathcal{K}^\alpha = (Y^{-1})^\alpha T$. Then

$$(Y^\alpha)'\mathcal{K}^\alpha + (Y^\alpha)(\mathcal{K}^\alpha)' = AY^\alpha(\alpha)\mathcal{K}^\alpha + Y^\alpha(\mathcal{K}^\alpha)A^*.$$

Since $(Y^\alpha)' = A(t)Y^\alpha$ it follows that

$$Y^\alpha(\mathcal{K}^\alpha)' = Y^\alpha \mathcal{K}^\alpha A^*.$$

Hence $(\mathcal{K}^\alpha)' = \mathcal{K}^\alpha B$ or $(\mathcal{K}^\alpha)' = A^*\mathcal{K}^*$.

Thus $\mathcal{K}^{*\alpha} = (Y^\alpha)C$ for some constant square matrix C.

Thus $T^\alpha(t) = Y^\alpha(t)\mathcal{K}^\alpha = Y^\alpha(t)C^*Y^{*\alpha}(t)$ (Take $C = C^*$).

Theorem 2.3: Let $T^\alpha(t)$ be any solution of (1.1) and $\bar{T}^\alpha(t)$ be a particular solution of (1.1), then

$$T^\alpha(t) = \bar{T}^\alpha(t) + Y^\alpha(t)C^*Y^{*\alpha}(t).$$

Proof: It can easily be verified that $T^\alpha(t) - \bar{T}^\alpha(t)$ is a solution of (1.2) and hence

$$T^\alpha(t) - \bar{T}^\alpha(t) = Y^\alpha(t)C^*Y^{*\alpha}(t)$$

and hence $T^\alpha(t) = \bar{T}^\alpha(t) + Y^\alpha(t)C^*Y^{*\alpha}(t)$.

Theorem 2.4: A particular solution $\bar{T}^\alpha(t)$ of (1.1) is given by

$\bar{T}^{\alpha}(t) = Y^{\alpha}(t)\int_a^t (Y^{-1})^{\alpha}(s)F(s)(Y^{*-1})^{\alpha}(s)ds.$

Proof: It can easily verified that $Y^{\alpha}(t)CY^{*\alpha}(t)$ is a solution of (1.2). such a solution cannot be a solution of (1.1) unless $F(t) = 0$ we seek a particular solution of (1.1) in the form

$\bar{T}^{\alpha}(t) = Y^{\alpha}(t)C(t)Y^{*\alpha}(t)$

$(Y^{\alpha})'CY^{*\alpha} + Y^{\alpha}C'Y^{*\alpha} + Y^{*}C(Y^{*\alpha})' = A(t)Y^{\alpha}(t)C(t)Y^{\alpha}(t) + Y^{\alpha}(t)C(t)Y^{\alpha}(t)A^{*}(t) + F(t)$

Since $(Y^{\alpha})' = A(t)Y^{\alpha}(t)$ and $(Y^{*})' = Y^{*\alpha}(t)A^{*}(t)$ it follows that

$Y^{\alpha}C'Y^{*\alpha} = F(t)$

Or

$C'(t) = (Y^{-1})^{\alpha}(t)F(t)(Y^{*-1})^{\alpha}(t)$

$C(t) = \int_a^t (Y^{-1})^{\alpha}(s)F(s)(Y^{*-1})^{\alpha}(s)ds$

Or

$$\bar{T}^{\alpha}(t) = Y^{\alpha}(t)C(t)Y^{*\alpha}(t) = Y^{\alpha}(t)\left[\int_a^t (Y^{-1})^{\alpha}(s)F(s)(Y^{*-1})^{\alpha}(s)ds\right]Y^{*\alpha}(t).$$

Hence

$$T^{\alpha}(t) = Y^{\alpha}(t)C(t)Y^{*\alpha}(t) = Y^{\alpha}(t)\left[\int_a^t (Y^{-1})^{\alpha}(s)F(s)(Y^{*-1})^{\alpha}(s)ds\right]Y^{*\alpha}(t).$$

Now, we consider the following systems

$$(T^{\alpha})'(t) = A(t)(T^{\alpha}(t) \quad (2.1)$$

and

$$(T_1^{\alpha})'(t) = T_1^{\alpha}(t)B(t), \quad (2.2)$$

where T^{α} and T_1^{α} are square matrices of order n. Let $Y^{\alpha}(t)$ and $Y_1^{\alpha}(t)$ be fundamental matrix solutions of (2.1), (2.2) respectively for each $\alpha \in [0,1]$. Then (2.1) and 2.2 can conveniently put in the form

$$[T^{\alpha}(t)\otimes T_1^{\alpha}(t)]' = [A(t)\otimes I_n) + \big(I_n\otimes B(t)\big)[(T^{\alpha}\otimes T_1^{\alpha})(t)]. \quad (2.3)$$

The following theorem will be of immense use for our future discussion.

Theorem 2.5: Suppose $(Y^{\alpha}\otimes Y_1^{\alpha})(t)$ be a fundamental matrix solution of (2.3). Then it is claimed that $Y^{\alpha}(t)$ and $Y_1^{\alpha}(t)$ are fundamental matrix solutions of (2.1) and (2.2) respectively.

For

$$(Y^{\alpha}\otimes Y_1^{\alpha})'(t) = [A(t)\otimes I_n) + \big(I_n\otimes B(t)\big)[(Y^{\alpha}\otimes Y_1^{\alpha})(t)]$$

$$[Y^{\alpha\prime}(t) - A(t)Y^{\alpha}(t)]\otimes Y_1^{\alpha}(t) = Y^{\alpha}(t) \bigotimes [B\, Y_1^{\alpha}(t) - Y_1^{\alpha\prime}(t)].$$

Multiplying both sides of the above relation by

$(Y^{\alpha})^{-1}(t)\otimes(Y_1^{\alpha})^{-1}$, We get

$$(Y^\alpha)^{-1}(t)[(Y^\alpha)'(t) - A(t)Y^\alpha(t)]\otimes I_n = I_n \otimes Y_1^{\alpha^{-1}}(t)\left[B\, Y_1^\alpha(t) - Y_1^{\alpha\prime}(t)\right]. \quad (2.4)$$

Equation (2.4) holds if and only if each side of (2.4) is equal to either a null matrix or a unit matrix.

Let us suppose

$$(Y^\alpha)^{-1}(t)[(Y^\alpha)'(t) - A(t)Y^\alpha(t)] = 0.$$

Then $(Y^\alpha)'(t) - A(t)Y^\alpha(t) = 0$. Then $Y^\alpha(t)$ is a fundamental matrix solution of (2.1) and suppose

$(Y^\alpha)^{-1}(t)[(Y^\alpha)'(t) - A(t)Y^\alpha(t)] = I_n$. Then

$$(Y^\alpha)'(t) - A(t)Y^\alpha(t) = Y^\alpha(t)$$

$$(Y^\alpha)'(t) = [A(t) + I_n]Y^\alpha(t),$$

which is a contradiction. Similar arguments holds for the R.H.S of (2.4). Conversely, $Y^\alpha(t)$ and $Y_1^\alpha(t)$ are fundamental matrix solutions of (2.1) and (2.2) respectively. Then it is claimed that $(Y^\alpha \otimes Y_1^\alpha)'(t)$ is a fundamental matrix solution of (2.3). Since for each $\alpha \in [0,1]$, we have

$(Y^\alpha)'(t) = A(t)Y^\alpha(t)$ and $(Y_1^\alpha)'(t) = B(t)\ Y_1^\alpha(t)$. Then

$$(Y^\alpha(t)\otimes Y_1^\alpha(t))' = (Y^{\alpha\prime}(t)\otimes Y_1^\alpha(t)) + (Y^\alpha(t)\otimes Y_1^{\alpha\prime}(t))$$

$$= (A(t)Y^\alpha(t)\otimes Y_1^\alpha(t)) + (Y^\alpha(t)\otimes B(t)Y_1^\alpha(t))$$

$$= (A(t)\otimes I_n)(Y^\alpha\otimes Y_1^\alpha)(t) + (I_n\otimes B(t))(Y_1^\alpha\otimes Y^\alpha)(t).$$

Hence the claim. Similar result hold for Lyapunov system (1.2)

3. Main Result

In this section, we establish our main result on the existence of (Ψ, Ψ^*) bounded solution of the linear Lyapunov system (1.1) and deduce stability criteria for the system (1.1). We first note the following theorem

Theorem 3.1: Let $u \in R^n$, then

(i) $[u]^\alpha \in P_k(t)$ for all $\alpha \in [0,1]$
(ii) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for all $0 \le \alpha_1 \le \alpha_2 \le 1$ and
(iii) If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then $[u]^\alpha = \bigcap_{k\ge1}[u]^{\alpha_k}$.

Conversely, if $\{A^\alpha : 0 \le \alpha \le 1\}$ is a family of subsets of R^n satisfying the above three conditions, then there exists a $u \in R^n$ such that

$$[u]^\alpha = A^\alpha \text{ for } 0 \le \alpha \le 1$$

and

$$[u]^0 = u_0 A^\alpha \subset A^0.$$

Lemma 3.1: For any $\alpha \in [0,1]$ such that $0 \le \alpha_1 \le \alpha_2 \le 1$ and for any $t \in R$

$$[\hat{u}(t)]^{\alpha_2} \subset [\hat{u}(t)]^{\alpha_1}.$$

Proof: Let $0 \le \alpha_1 \le \alpha_2 \le 1$. Then by hypothesis, it follows that $[\hat{u}(t)]^{\alpha_2}$ is contained in $[\hat{u}(t)]^{\alpha_1}$ and hence

$\hat{u}^{\alpha_2}(t) = u_1^{\alpha_2}(t) \times u_2^{\alpha_2}(t) \times \ldots \times u_s^{\alpha_2}(t) \subset u_1^{\alpha_1}(t) \times u_2^{\alpha_1}(t) \times \ldots \times u_s^{\alpha_1}(t)$

$= \hat{u}^{\alpha_1}(t).$

Thus, we have the selection of inclusions $S'\hat{u}^{\alpha_2}(t) \subset S'\hat{u}^{\alpha_1}(t)$ and hence

$\hat{T}(t) \in A(t)\hat{T}(t) + \hat{T}(t)A^*(t)$

$\in Y^\alpha(t)CY^{*^\alpha}(t),$

and further

$$Y^{\alpha_1}(t)CY^{*^{\alpha_1}}(t) \subset Y^{\alpha_2}(t)CY^{*^{\alpha_2}}(t)$$

Thus $\hat{T}^{\alpha_1}(t) \subset \hat{T}^{\alpha_2}(t)$. Further

$$T^{\alpha_1}(t) \in A(t)T^{\alpha_1}(t) + T^{\alpha_1}(t)A^*(t) + F(t)$$

and

$$T^{\alpha_2}(t) \in A(t)T^{\alpha_2}(t) + T^{\alpha_2}(t)A^*(t) + F(t).$$

It follows that $\hat{T}^{\alpha_2}(t) \subset \hat{T}^{\alpha_1}(t)$.

Let Y^α be a fundamental matrix solution of (2.1). Then it can easily be verifies that $Y^\alpha(t)CY^{*^\alpha}(t)$ is a solution of the homogeneous Lyapunov system (1.2). Further, let the vector space R^n be represented as direct sum of three subspaces say X_-^α, X_0^α, X_+^α such that a solution of (1.2) is Ψ^α-bounded on R if and only if $T(0) \in T_0$ and Ψ bounded on $R_+ = [0,\infty)$ if and only if $T^\alpha(0) \in X_-^\alpha \otimes X_0^\alpha$. Also let $P_-^\alpha, P_0^\alpha, P_+^\alpha$ denote the corresponding projection on R^n onto $X_-^\alpha, X_0^\alpha, X_+^\alpha$ respectively for each $\alpha \in [0,1]$. We are now in position to prove the following result.

Theorem 3.2: Let $A(t)$ be a continuous $(n \times n)$ matrix on R. Then the system (1.2) has at least one Ψ^α bounded solution on R for every continuous and Ψ^α-bounded function

$F: R \to R^{n\times n}$ if and only if there exists a constant $K > 0$ such that

$$\int_{-\infty}^{t} \left\| \Psi^\alpha(t)Y^\alpha(t)Y^{*\alpha}(t)\Psi^{*\alpha}(t)P_-\Psi^{*\alpha^{-1}}(s)Y^{*^{-1^\alpha}}(s)Y^{-1^\alpha}(s)\Psi^{-1^\alpha}(s) \right\| ds +$$

$$\int_{t}^{0} \left\| \Psi^\alpha(t)Y^\alpha(t)Y^{*\alpha}(t)\Psi^{*\alpha}(t)(P_0 + P_+)(\Psi^{*^{-1}})^\alpha(s)(Y^{*^{-1}})^\alpha(s)(Y^{-1})^\alpha(s)(\Psi^{-1})^\alpha(s) \right\| ds +$$

$$\int_0^\infty \left\| \Psi^\alpha(t)Y^\alpha(t)Y^{*\alpha}(t)\Psi^{*\alpha}(t)P_+(\Psi^{*^{-1}})^\alpha(s)(Y^{*^{-1}})^\alpha(s)(Y^{-1})^\alpha(s)(\Psi^{-1})^\alpha(s) \right\| ds \le K, \text{ for } t \ge 0 \quad (3.1)$$

and

$$\int_{-\infty}^{0} \left\| \Psi^\alpha(t)Y^\alpha(t)Y^{*\alpha}(t)\Psi^{*\alpha}(t)P_-(\Psi^{*^{-1}})^\alpha(s)(Y^{*^{-1}})^\alpha(s)(Y^{-1})^\alpha(s)(\Psi^{-1})^\alpha(s) \right\| ds +$$

$$\int_{0}^{\infty} \left\| \Psi^\alpha(t)Y^\alpha(t)Y^{*\alpha}(t)\Psi^{*\alpha}(t)(P_0 + P_+)(\Psi^{*^{-1}})^\alpha(s)(Y^{*^{-1}})^\alpha(s)(Y^{-1})^\alpha(s)(\Psi^{-1})^\alpha(s) \right\| ds +$$

$$\int_{t}^{\infty} \left\| \Psi^\alpha(t)Y^\alpha(t)Y^{*\alpha}(t)\Psi^{*\alpha}(t)P_+(\Psi^{*^{-1}})^\alpha(s)(Y^{*^{-1}})^\alpha(s)(Y^{-1})^\alpha(s)(\Psi^{-1})^\alpha(s) \right\| ds \le K$$

for $t \ge 0$. (3.2)

Proof: To make proof simple, we use the following notation

$\Psi^{\alpha}(t)\Upsilon^{\alpha}(t) = \Phi^{\alpha}(t)$ and $Y^{*\alpha}(t)\Psi^{*\alpha}(t) = \Phi^{*^{\alpha}}(t)$ for each $\alpha \in [0,1]$. First suppose that the Lyapunov linear system (1.1) has atleast one Ψ^{α} −bounded solution on R for every $\alpha \in [0,1]$. Let B denote the set of all Ψ^{α} −bounded solutions and $T: R \to R^{n\times n}$ with the norm

$$\|T^{\alpha}\|_B = Sup_{t\in R}\|\Psi^{\alpha}(t)T^{\alpha}(t)\|,$$

and let D denote the set of all Ψ^{α} −bounded and continuously differentiable functions

$T^{\alpha}: R \to R^{n\times n}$ such that $T^{\alpha}(0) \in X_- \oplus X_+$ and

$T^{\alpha} - A(t)T^{\alpha} - T^{\alpha}B(t) \in B$. Clearly, D is a vector space and if we define norm on D by

$\|T^{\alpha}\|_D = \|T^{\alpha}\|_B + \left\|T^{\alpha\prime} - A(t)T^{\alpha}(t) - T^{\alpha}(t)A^*(t)\right\|_B.$

Then it can easily be proved that $(D, \|.\|_D)$ is a Banach space.

For, let $\{T_n^{\alpha}\}_{n\in N}$ be a fundamental sequence of elements of D. Then, $(T_n^{\alpha})_{n\in N}$ is a fundamental sequence in B. Therefore, there exists a continuous and Ψ^{α}-bounded function for each $\alpha \in [0,1]$ such that $\lim_{n\to\infty} \Psi^{\alpha}(t)T_n^{\alpha}(t)$ converges to $\Psi^{\alpha}(t)\, T_n^{\alpha}(t)$ uniformly on R. From the inequality

$\|T_n^{\alpha}(t) - T^{\alpha}(t)\| \le \|(\Psi^{-1})^{\alpha}(t)\|\|\Psi^{\alpha}(t)T_n^{\alpha}(t) - \Psi^{\alpha}(t)T^{\alpha}(t)\|.$

It follows that $\lim_{n\to\infty} T_n^{\alpha}(t) = T^{\alpha}(t)$ for each $\alpha \in [0,1]$ uniformly on every compact subset of

R. Thus

$T^{\alpha}(0) \in X_- \oplus X_+.$

Similarly,

$\lim_{n\to\infty}\left(T_n{}^{\alpha\prime}(t) - T_n(t)A^*(t) - A(t)T_n(t)\right) = F(t)$, uniformly on every compact subset of R and hence

$\lim_{n\to\infty} \Psi^{\alpha}(t)T_n^{\alpha}(t) = \Psi^{\alpha}(t)T^{\alpha}(t), t \in R, \alpha \in [0,1]$.

Hence.

$\lim_{n\to\infty}\|T_n^{\alpha}(t) - T^{\alpha}(t)\| = 0$. This proves that $(D, \|.\|_D)$ is a Banach space.

Differentiable functions $T^{\alpha}: R \to R^{n\times n}$ such that $T^{\alpha}(0) \in X_-\oplus X_+$ and

$T^{\alpha} - A(t)T^{\alpha} - T^{\alpha}A^*(t) \in B$. Clearly, D is a vector space and if we define norm on D by $\|T^{\alpha}\|_D = \|T^{\alpha}\|_B + \left\|T^{\alpha\prime} - A(t)T^{\alpha}(t) - T^{\alpha}(t)A^*(t)\right\|_B$

Then, it can easily be prove that $(D, \|.\|_D)$ is a Banach space. For, let $\{T_n^{\alpha}\}_{n\in N}$,

$\alpha \in [0,1]$ be a fundamental sequence of elements of D. Then $\{T_n^{\alpha}\}_{n\in N}$ is a fundamental sequence in B. Therefore, there exists a continuous Ψ^{α}-bounded function for each $\alpha \in [0,1]$ such that $\lim_{n\to\infty}\|\Psi^{\alpha}(t)T_n^{\alpha}(t)\|$ converges to $\Psi^{\alpha}(t)T^{\alpha}(t)$ uniformly on R.

Thus, $T^{\alpha}(0) \in X_-\oplus X_+$ for each $\alpha \in [0,1]$.

Similarly,

$$\lim_{n\to\infty}\left((T_n^{\alpha})'(t) - A(t)T_n^{\alpha}(t) - T_n^{\alpha}(t)A^*(t)\right) = F(t).$$

Uniformly on every compact subset of R and hence

$$\lim_{n\to\infty} \Psi^{\alpha}(t)T_n^{\alpha}(t) = \Psi^{\alpha}(t)T^{\alpha}(t) \text{ for } t \in R \text{ and } \alpha \in [0,1].$$

Hence

$$\lim_{n\to\infty}\|T_n^\alpha(t) - T^\alpha(t)\| = 0 \text{ for each } \alpha \in [0,1].$$

This proves that $(D, \|.\|_D)$ is a Banach space.

Step 2: There exists a positive constant say K_0 such that for every $F \in B$ and for the corresponding solution $T \in D$ of Lyapunov system (1.1), we have

$$Sup_{t\in R}\|\Psi(t)T(t)\| \leq K_0\ Sup_{t\in R}\|\Psi(t)F(t)\|. \quad (3.3)$$

For, define the mapping $X: D \to B$ such that

$$X\,T^\alpha = T^{\alpha\prime} - A\,T^\alpha - T^\alpha A^*.$$

This mapping us linear and bounded with $\|X\| \leq 1$.

Let XT = 0, then T' = A $T + TA^*, T \in D$.

This shows that X is a Ψ bounded solution on R on (1.2). Then $X(0) \in X_0 \cap (X_- \oplus X_+) = \{0\}$.

Thus T=0, such that X is one-one and can easily be proved that T is onto.

Then $u = T^\alpha - \overline{T^\alpha}$ is a solution of (1.2) with $u(0) - (P_- + P_+)u(0) = P_0 u(0)$

From the definition of X_0, it follows that u is Ψ bounded on R. Then $\bar{T}$ belongs to D and

$u\bar{T} = F$.

From a fundamental lemma of Banach space, it follows that

$\|u^{-1}F\|_D \leq \|u^{-1}\|\|F\|_D$ for all $F \in B$ and hence step 2 follows

Step 3: Let $T_1 < 0 < T_2$ be fixed points and let $F: R \to R^{n^2}$ be continuous and Ψ-bounded functions which vanishes on $(-\infty, \pi) \cup [T_2, \infty)$. Then it is easily to see that the mapping $T: R \to R^{n^2}$ defined by

$$T(t) = \begin{cases} -\int_{T_0}^{T_1} \Phi^\alpha(t)\Phi^{*\alpha}(t)P_+(\Phi^{*\alpha})^{-1}(s)(\Phi^{-1})^\alpha(s)F(s)ds \\ -\int_{T_1}^{T_2} \Phi^\alpha(t)\Phi^{*\alpha}(t)P_+\left(\Phi^{*-1}\right)^\alpha(s)(\Phi^{-1})^\alpha(s)F(s)ds \ for\ t \leq T_1 \\ -\int_{T_1}^{t} \Phi^\alpha(t)\,(\Phi^{*\alpha})^{-1}(t)P_-\left(\Phi^{-1\alpha}\right)^*(s)(\Phi^\alpha)^{-1}(s)F(s)ds \\ +\int_0^t \Phi^\alpha(t)\,(\Phi^{*\alpha})^{-1}(t)P_0\left(\Phi^{-1*}\right)^\alpha(s)(\Phi^\alpha)^{-1}(s)F(s)ds \ for\ T_1 \leq t \leq T_2 \\ -\int_t^{T_2} \Phi^\alpha(t)\Phi^{*\alpha}(s)P_+\left(\Phi^{*-1}\right)^\alpha(s)(\Phi^\alpha)^{-1}(s)F(s)ds + \\ \int_{T_1}^{T_2} \Phi^\alpha(t)\Phi^{*\alpha}(t)P_-\left(\Phi^{*-1}\right)^\alpha(s)(\Phi^{-1})^\alpha(s)F(s)ds + \\ \int_0^T \Phi^\alpha(t)\Phi^{\alpha*}(t)\,P_0(\Phi^{*\alpha})^{-1}(s)(\Phi^\alpha)^{-1}(s)F(s)ds \ for\ t > T_2, \end{cases}$$

is the solution of D of Lyapunov system (1.1)

Similarly, the other case follows. If we put

$$G(t,s)=\begin{cases}\Phi^{\alpha}(t)\Phi^{*\alpha}(t)P_{-}(\Phi^{*-1})(s)(\Phi^{-1})^{\alpha}(s), t>0, s\le 0\\ \Phi^{\alpha}(t)\Phi^{*\alpha}(t)(P_0+P_{-})\left(\Phi^{*-1}\right)^{\alpha}(s)(\Phi^{-1})^{\alpha}(s), t>0, s>0, s<t\\ -\Phi^{\alpha}(t)\Phi^{*\alpha}(t)P_{+}(\Phi^{*-1})(s)(\Phi^{-1})^{\alpha}(s), t>0, s>0, s\le t\\ \Phi^{\alpha}(t)\Phi^{*\alpha}(t)P_{-}(\Phi^{*-1})(s)(\Phi^{-1})^{\alpha}(s), t\le 0, s<t\\ -\Phi^{\alpha}(t)\Phi^{*\alpha}(t)(P_0+P_{+})(\Phi^{*-1})(s)(\Phi^{-1})^{\alpha}(s), t\le 0, s\ge t, s<0\\ -\Phi^{\alpha}(t)\Phi^{*\alpha}(t)P_{+}(\Phi^{*-1})(s)(\Phi^{-1})^{\alpha}(s), t\le 0, s\ge t, s\ge 0,\end{cases}$$

we then have $T(t)=\int_{T_1}^{T_2}G^{\alpha}(t,s)F(s)ds, t\in R$ and for each $\alpha\in[0,1]$.

Now, consider for $t>T_2$,

$\int_{T_1}^{T_2}G^{\alpha}(t,s)F(s)ds=\int_{T_1}^{0}G^{\alpha}(t,s)F(s)ds+\int_{0}^{T_2}G^{\alpha}(t,s)F(s)ds$

$=\int_{T_1}^{T_2}\Phi^{\alpha}(t)\Phi^{*\alpha}(t)P_{-}\left(\Phi^{*-1}\right)^{\alpha}(s)(\Phi^{-1})^{\alpha}(s)F(s)ds$

$+\int_{0}^{T_2}\Phi^{\alpha}(t)\Phi^{*\alpha}(t)(P_0+P_{-})\left(\Phi^{*-1}\right)^{\alpha}(s)(\Phi^{-1})^{\alpha}(s)F(s)ds$

$=\int_{T_1}^{T_2}\Phi^{\alpha}(t)\Phi^{*\alpha}(t)P_{-}\left(\Phi^{*-1}\right)^{\alpha}(s)(\Phi^{-1})^{\alpha}(s)F(s)ds$

$+\int_{0}^{T_2}\Phi^{\alpha}(t)\Phi^{*\alpha}(t)P_0\left(\Phi^{*-1}\right)^{\alpha}(s)(\Phi^{-1})^{\alpha}(s)F(s)ds$

$=T^{\alpha}(t)$.

Similarly, the other cases follow.

Now, to prove the other part, suppose the fundamental matrix solution of $Y^{\alpha}(t)$ of (2.1) satisfies either condition (3.1) or (3.2) for some $K>0$. Then it can easily be proved that there exists a $\Psi^{\alpha}(t)$ bounded solution on R of (1.1).

Theorem 3.2: Suppose that the homogeneous Lyapunov system has no non-trivial Ψ^{α}- bounded solution on R for each $\alpha\in[0,1]$ and suppose that the fundamental matrix solution satisfies the condition

$$\int_{-\infty}^{t}\left\|\Phi^{\alpha}(t)\Phi^{*\alpha}(t)P_1\left(\Phi^{*-1}\right)^{\alpha}(s)(\Phi^{-1})^{\alpha}(s)\right\|ds+\int_{t}^{\infty}\left\|\Phi^{\alpha}(t)\Phi^{*\alpha}(t)P_1\left(\Phi^{*-1}\right)^{\alpha}(s)(\Phi^{-1})^{\alpha}(s)\right\|ds\le K,$$

and further assume that Ψ^{α}-bounded function $F: R\to R^{n\times n}$ is such that $\lim_{t\to\infty}\|\Psi^{\alpha}F(t)\|=0$ for each $\alpha\in[0,1]$. Then the non-homogeneous Lyapunow system (1.1) has unique solution $T^{\alpha}(t)$ on R such that

$\lim_{t\to\infty}\|\Psi^{\alpha}F(t)\|=0$ for each $\alpha\in[0,1]$.

Note that if $\Psi^{\alpha}(t)$ is a fundamental matrix solution of (2.1) if and only if $\Psi^{*\alpha}(t)$ is a fundamental matrix solution of (2.2) with $B(t)=A^{*}(t)$. If the system (2.1) has a $\Psi^{\alpha}(t)$-bounded solution, then system (2.2) has $\Psi^{*\alpha}$-bounded solution for $t\in R$ and $\alpha\in[0,1]$. If the system (2.1) is stable then (2.2) is a stable with $B=A^{*}$.

References:

1. O. Kaleva, "Fuzzy differential equations, Fuzzy sets and systems", 24 p: 301-317 (1987)
2. Kasi Viswanadh V. Kanuri, K. N. Murty, "Three-Point boundary value problems associated with first order matrix difference system-existence and uniqueness via shortest and closest Lattice vector methods", Journal of Nonlinear Sciences and Applications, Volume 12, Issue 11, (2019) 720--727
3. Kasi Viswanadh V. Kanuri, Existence of Ψ^{α}-bounded solutions for fuzzy dynamical systems on time scales", International Journal of Scientific & Engineering Research, 2020, Vol. 11, No. 5, 613--624.
4. Kasi Viswanadh V. Kanuri, R. Suryanarayana, K. N. Murty, Existence of Ψ-bounded solutions for linear differential systems on time scales, Journal of Mathematics and Computer Science, 20 (2020), no. 1, 1--13.

5. Murty, K. N., Andreou, S., Viswanadh, K. V. K., Qualitative properties of general first order matrix difference systems. Nonlinear Studies, 16(2009), no. 4, 359--370.
6. K. N. Murty, V. V. S. S. Balaram, K.V. K. Viswanadh, "Solution of Kronecker Product Initial Value Problems Associated with First Order Difference System via Tensor-based Hardness of the Shortest Vector Problem", Electronic Modeling, vol. 6, p: 19--33 (2008).
7. K.N. Murty, K.V.K. Viswanadh, P. Ramesh, Yan Wu, "Qualitative properties of a system of differential equation involving Kronecker product of matrices", Nonlinear Studies, Vol 20, No: 3, P: 459--467 (2013)
8. K.N. Murty, Yan Wu, Viswanadh V. Kanuri, "Metrics that suit for dichotomy, well conditioning of object oriented design", Nonlinear Studies, Vol 18, No: 4, P: 621--637 (2013)
9. Kasi Viswanadh V Kanuri, Y. Wu, K.N. Murty, "Existence of ($\Phi \otimes \Psi$) bounded solution of linear first order Kronecker product of system of differential equations", International Journal of Science and Engineering Research, 2020, Vol 11, No. 6, P: 721--730.
10. Viswanadh V. K, K.N. Murty, P. Sailaja, "A new approach to the construction of a transition matrix and applications to control systems", Proc. of the International Conference on Material science, Dec 2019 (India)
11. Martin Bohner, Allan Peterson, "Dynamical equations on time scales", Birkhauser, Boston (2003)
12. V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, "Dynamical systems on measure chains", Kluwer Academic publishers, Dordrecht (1998).
13. V Lakshmikantham, K.N. Murty and J. Turner, "Two-Point boundary value problems associated with non-linear fuzzy differential equations", Mathematical inequalities and applications, Vol 4, P: 527-533 (2001).
14. Charyulu L. N. Rompicherla, Sundaranand V. Putcha, Deekshitulu G.V.S.R, Existence of ($\Phi \otimes \Psi$) bounded solutions for linear first order kronecker product systems, International Journal of Scientific Research, Vol 11, No: 6, P: 39047-39053 (2020)
15. Kasi Viswanadh V. Kanuri, SriRam, Bhagavathula and K.N. Murty, "New Stability Analysis of Linear First order Sylvester system", International Journal of Engineering and Computer Science., Vol 11, P: 25252-25259 (2020)
16. Madan L Puri, Dan A Ralescu, Fuzzy random variables, Journal of Mathematical Analysis and Applications, Volume 114, Issue 2, 1986, Pages 409-422.
17. Y. Wu, D.L. Nethi, K.N.Murty, Initial Value Problems Associated With First Order Fuzzy Difference System -Existence And Uniqueness, International Journal of Recent Scientific Research. 11 (2020), no. 3, 37846--37848.
18. Lakshmikantham, V. and Mohapatra, R. Theory of fuzzy differential equations and inclusions, Taylor - Francis, 2003.
19. Phil Diamond, Peter E. Kloeden, Metric Spaces of Fuzzy Sets: Theory and Applications, World Scientific, 1994.