CONGRUENCE RELATION ON TOTALLY ORDERED TERNARY SEMIRINGS

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Abstract*: In this paper we discussed about the congruence relation'□' in a totally ordered ternary semiring (T,+,.,≤) defined by a □ b □ a^3 = aba =bab for all a,b in T. We characterize some of the properties in a totally ordered ternary semirings with congruence relation like R-Commutative, L-Commutative, Rectangular band, Anti-regular etc.*

Keywords: Totally ordered ternary semirings, R-Commutative, -Commutative, Anti-Regular, quasi-seperative.

1. INTRODUCTION: The algebraic theory of semigroups was widely studied by Clifford [1,2],and PERTICH[12] .The theory of ternary algebraic systems was introduced by LEHMER[3] in 1932.LEHMER investigated certain algebraic systems called triplexes which turned out to be commutative ternary groups .But earlier these structures are studied by PRUFER[11] in 1924,BAER in 1929.Generalizing the notion of ternary ring introduced by LISTER[4]. DUTTA [5]and KAR [5] introduced the notion of ternary semirings. On other hand there is considerable impact of semigroup theory and semiring theory in the development of ordered semirings both theory and applications. In this direction the works of M.Satyanarayana [7], J. Hanumantachari [9], P.J.Weirnet [9], K. Venuraju [9], Jonathan S.Golan [6] are worth mentioning. In this paper we want to study the importance of congruence relation in totally ordered ternary semiring.

2. Preliminaries: In this section we have some basic definitions & results used in this paper

Definition2.1: A system (T,≤) is called partially ordered set if it satisfies the following axioms on T

(i) a ≤ a (Reflexivity)

(ii) a ≤ b, b ≤ a ⟹ a = b (Anti symmetry)

(iii) a≤ b, b ≤ c ⟹ a ≤ c (Transitivity)

for all a, b, c ∈ T

Definition 2.2: A Ternary semigroup (T,.) is said to be a partially ordered ternary semigroup if T is partially ordered set such that for any a,b ∈ T if a ≤ b ⟹ [cad] ≤[cbd], [acd] ≤ [bcd] ,[cda] ≤ [cdb] for c,d in T.

Definition 2.3: A Ternary semiring T is called Partial ordered ternary semiring if there exist partial order '≤' on T such that

(i)(T,+, ≤) is partially ordered semigroup

(ii) (T,., ≤) is partially ordered semigroup. It is denoted by (T,+,. ≤)

Example2.4: Let T ={0,a,b,c} be a partially ordered ternary semiring with respect to binary operation '+' and ternary multiplication'.' where a<b<c<0 and +,. are defined as follows

.	a	b	c	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	0
0	0	0	0	0

+	a	b	c	0
a	a	b	c	0
b	b	b	c	0
c	c	c	c	0
0	0	0	0	0

Definition 2.5: A ternary semigroup (T,.) is a band if every element in T is an idempotent .i.e a^3= a for all a ∈ T.

Definition 2.6: A commutative band is called semilattice.

Definition 2.7: A ternary semigroup (T,.) is called rectangular band if ababa=a for all a,b in T.

Definition 2.8: A ternary semigroup (T,.) is called quasi -seperative if a^3 = aba =bab = b^3 implies that a = b for all a,b in T.

Definition 2.9: A ternary semigroup (T,.) is said to be left singular if it satisfies the identity ab^2 = a for all a,b in T.

Definition 2.10: A ternary semigroup (T,.) is said to be lateral singular if it satisfies the identity bab = a for all a,b in T.

Definition 2.11: A ternary semigroup (T,.) is said to be right singular if it satisfies the identity b^2a = a for all a,b in T.

Definition 2.12: A ternary semigroup (T,.) is said to be two sided singular, if it is both left and right singular.

Definition 2.13:A ternary semigroup (T,.) is said to be singular if it is left, lateral and right singular.

Definition 2.14: A ternary semigroup (T,.) is said to be left regular, if it satisfies the identity $a = a^3xy$ for all a,x,y in T.

Definition2.15: A ternary semigroup (T,.) is said to be right regular ,if it satisfies the identity $a = xya^3$ for all a, x,y in T.

Definition 2.16: A ternary semigroup (T,.) is said to be lateral regular, if it satisfies the identity $a = x a^3 y$ for all a, x, y in T.

Definition 2.17: A ternary semigroup (T,.) is said to be two sided regular if it is left as well as right regular.

Definition 2.18: A ternary semigroup (T,.) is said to be regular if it is left, right and lateral regular.

Definition2.19: A system $(T,\leq)$ where the relation '$\leq$' on T satisfying the following axioms

(i)Reflexivity: $a \leq a$

(ii) Anti symmetry: $a \leq b$,$b \leq a$ imply $a = b$

(iii) Transitivity $a \leq b$, $b \leq c$ imply $a \leq c$

(iv) Linearity $a \leq b$ (or) $b \leq a$

For all a,b,c in T is called a totally ordered set.

Definition 2.20: A ternary semiring (T,+,.) is said to be totally ordered ternary semiring if there exists a partially order "$\leq$" on T such that

(i)$(T,+,\leq)$ is a totally order semigroup

(ii)$(T,.,\leq)$ is totally ordered semigroup and it is denoted by $(T,+,.,\leq)$

EXAMPLE2.21: Consider the set T={1,2,3,4} with the order 1 <2 <3 <4 and with the following binary operation '+' and Ternary multiplication '.'

+	1	2	3	4
1	2	3	4	4
2	3	4	4	4
3	4	4	4	4
4	4	4	4	4

.	1	2	3	4
1	2	4	4	4
2	4	4	4	4
3	4	4	4	4
4	4	4	4	4

Result 2.22[8]: A semigroup S is a totally ordered with respect to its partial order if and only if S is one of the following:

(i)S=E(S) and $(E(S),\leq)$ is a chain.

(ii)S=E(S) U {a} for some a is not belongs to E(S) such that ea=ae binary operation '+' and Ternary multiplication '.'

=e for every 'e' in E(S) and E(S) is a chain with greatest element a^2.

3. CONGRUENCE RELATION ON TOTALLY ORDERED TERNARY SEMIRING:

Theorem3.1: Assume that (T,+,.) is a ternary semiring in which(T,.) is semilattice such that a relation ρ on T defined by $a\rho b \Leftrightarrow a^3=aba=bab$ for all a,b in T. If (T,.) is Singular then $(T,+,.,\leq)$ is partial ordered ternary semiring.

Proof: As given (T,+,.) is a ternary semiring in which (T,.) is semilattice such that 'ρ' is a relation on T defined by $a\rho b \Leftrightarrow a^3=aba=bab$ for all a,b in T.

Reflexive: Consider b = a

Then $aba = a\,a\,a = a^3$

$bab = a\,a\,a = a^3$

$\Longrightarrow a^3 = aba = bab$

$\Longrightarrow a\,\rho a$

Therefore 'ρ' is reflexive.

Antisymmetric:

Let $a\rho b \Leftrightarrow$ a3=aba=bab

$b\rho a \Leftrightarrow b^3=bab=aba$

$\Longrightarrow a^3 = aba = bab = b^3$

$\Longrightarrow a^3 = b^3$

$\Longrightarrow a = b$ (Since (T,.) is band)

Therefore 'ρ' is antisymmetric.

Transitive:

Let $a\rho b \Leftrightarrow a^3=aba=bab$

$b\rho c \Leftrightarrow b^3=bcb=cbc$

$a^3 = aba$

$= ab^3a$

$= a(bcb)a$

$= aca$ (since (T,.) is singular)

$a^3 = aca$

Similarly we can prove $a^3 = cac$

$\Rightarrow a\rho c$

Therefore 'ρ' is transitive.

Compatibility

Let $a\rho b \Leftrightarrow a^3 = aba = bab$

$\Rightarrow a^3(cd)^3 = aba(cd)^3 = bab(cd)^3$

$\Rightarrow (acd)^3 = aba(cd)(cd)(cd) = bab(cd)(cd)(cd)$

$\Rightarrow (acd)^3 = a(ba(cd))(cd)(cd) = b(ab(cd))(cd)(cd)$

$\Rightarrow (acd)^3 = a((cd)\, ba)(cd)(cd) = b((cd)ab)(cd)(cd)$

(Since (T,.) is commutative)

$\Rightarrow (acd)^3 = (acd)(ba(cd))(cd) = (bcd)(ab(cd))(cd)$

$\Rightarrow (acd)^3 = (acd)(bcd)(acd) = (bcd)(acd)(bcd)$

$(acd)\ \rho\ (bcd)$

Similarly we can prove $(cad)\ \rho\ (cbd)$ and

$(cda)\ \rho\ (cdb)$

Let $a\ \rho\ b$ $a^3 = aba = bab$

$(a + c)^3 = (a + c)(a + c)(a + c)$

$= (a + c)(a^3 + c)(a + c)$

$= (a + c)(aba + c)(a + c)$

$= (a+c)(b+c)(a + c)$ (since (T,.) is singular)

Therefore $(a + c)\ \rho\ (b + c)$

Similarly we can prove $(c + a)\ \rho\ (c + b)$

Therefore (T,+,., ρ) is a partially ordered ternary semiring.

Theorem 3.2: Suppose (T,.) is a ternary semigroup which is semilattice. Define 'ρ' on T by $a\ \rho\ b \Leftrightarrow a^3 = aba = bab$ for all a,b in T .then (T,., ρ) is totally ordered ternary semigroup.

Proof: As given (T,.) is a ternary semigroup which is semilattice. Define 'ρ' on T by $a\ \rho\ b \Leftrightarrow a^3 = aba = bab$ for all a,b in T .

From the theorem 3.1, T is partially ordered ternary semigroup w.r.t p.o. relation ρ

Again from the result 2.22[8] we conclude that (T,., ρ) is totally ordered ternary semigroup.

Remark: (T,.,ρ) is Totally Ordered if (T,.) is semi lattice

Definition 3.3: A ternary semigroup (T,.) is called quasi commutative if for any a,b,c in T we have $abc = b^nac = bca = c^nba = cab = a^ncb$ for some positive integer 'n'.

Definition 3.4: A ternary semigroup (T,.) is said to Archimedean if for every a, b in T $a^n = xby$ for some natural number 'n'.

Definition3.5:A ternary semigroup (T,.) is said to be R-Commutative for all a,b,c in T and x,y in T^1 then $abc = xy(bac)$ where T^1 is ternary semigroup T with identity '1'.

Theorem 3.6: Assume that (T,., ρ) is a totally ordered ternary semigroup where ' ρ' defined on T such that $a\ \rho\ b \Leftrightarrow a^3 = aba = bab$ for all a,b in T .If (T,.) is R-commutative, Archimedean then it is quasi commutative

Proof: As given (T,., ρ) is a totally ordered ternary semigroup. where ' ρ' defined on T such that $a\ \rho\ b \Leftrightarrow a^3 = aba = bab$ for all a,b in T. (T,.) is R-commutative .i.e. $abc = xy(bac)$ and if (T,.) is Archimedean $b^n = xay$,$a^m = xby$ for all a,b in T and for some x,y in T.

Since (T,.) is totally ordered every two elements are comparable in T .i.e either $a\ \rho\ b$ or $b\ \rho\ a$ for any a , b in T

To prove T is quasi commutative consider the following cases

Case (i): consider $a\ \rho\ b \Leftrightarrow a^3 = aba = bab$

$b^nac = (xay)\, ac$

$= (xay)\, a^3c$

$= (xya)(aba)c$

$= xy(a\, a\, b\,)a\, c$

$= x\, y\, (\, b\, a\, a)\, a\, c$

$= x\, y\, b\, a^3\, c$

$= x\, y(b\, a\, c\,)$

$b^nac = abc$ (since (T,.) is R-commutative)

Similarly we can prove bca=c^nba=cab=a^ncb

Case (ii) : consider b ρ a ⇔ b^3 = bab = aba

a^mbc = (xby)bc

= (xby)b^3c

= (xby)(bab)c

= (xyb)(bab)c

= xy(b b a) b c

= xy (a b b) b c

= xy a b^3 c

= xy (abc)

a^mbc = bac

Similarly we can prove bca= c^mba = cab= a^mcb

Hence (T,.) is quasi commutative.

Definition 3.7: A ternary semigroup (T,.) is called L-commutative ternary semigroup if for every element a,b,c in T there exist x,y in T^1 such that abc =(acb)xy where T^1 is ternary semigroup with identity

Theorem 3.8: If (T,., ρ) is a totally ordered ternary semigroup where ' ρ' on T defined as a ρ b ⇔ a^3 =aba=bab for all a,b in T .If (T,.) is L, R-commutative then (T,.) is intra regular.

Proof: As given (T,., ρ) is a totally ordered ternary semigroup such that a ρ b ⇔ a^3 =aba=bab for all a,b in T and

(T,.) is L-commutative .i.e abc = (acb)xy

(T,.) is R-commutative .i.e abc = xy(bac) for all a,b ,c in T and x,y in T^1 where T^1 is a ternary semigroup with identity element '1'

Since T is totally ordered either a ρ b or b ρ a

To have (T,.) is intra regular Consider the following cases

Case(i): If a ρ b ⇔ a^3 =aba=bab

x a^5 y = x a a^3 a y

= x a a a y

= x a^3 y

= x y a^3

= xy(aba)

= b a a (since (T,.) is R-commutative)

= a b a

= a^3

x a^5 y = a

Case (ii): If b ρ a ⇔ b^3 =bab=aba

x b^5 y = x b b^3 b y

= x b b b y

= x b^3 y

= b^3 x y

= (bab)xy

= b b a(since (T,.) is L-commutative)

= b a b

= b^3

x b^5 y = b

Therefore (T,.) is intra regular.

Theorem 3.9: Let (T,.,ρ) be a totally ordered ternarysemigroup where ' ρ' on T defined by a ρ b ⇔ a^3 =aba=bab for all a,b in T .Then (T,.) is rectangular band.

Proof: Given that (T,., ρ) is a totally ordered ternary semigroup and a ρ b ⇔ a^3 =aba=bab for all a,b in T .we have to prove (T,.) is rectangular band .i.e. ababa= a for all a, b in T. For that consider the following cases

Case (i): If a ρ b ⇔ a^3 = aba = bab

ababa = ab(aba)

= aba^3

= aba

= a

ababa = a

Case (ii): b ρ a ⇔ b^3 = bab = aba

babab = (bab)ab

= b^3ab

= bab

= b^3

babab = b

Therefore (T,.) is rectangular band.

Definition 3.10: A ternary semigroup (T,.) is normal if x(abc)x = x(bac)x =x(cab)x =x(acb)x= x(bca)x = x(cba)x for all x,a,b in T.

Theorem 3.11: Suppose (T,., ρ) is a totally ordered ternary semigroup where ' ρ' on T defined by a ρ b ⇔ a^3 =aba=bab for all a,b in T .Then (T,.) is normal.

Proof: As given (T,., ρ) is a totally ordered ternary semigroup and a ρ b ⇔ a^3 =aba=bab for all a,b in T .We have to show (T,.) is normal. For that consider the following cases

Case (i): If a ρ b ⇔ a^3 = aba = bab

a(bac)a = (aba)ca

= a^3ca

= aca

= aca^3

= ac(aba)

a(bac)a = a(cab)a

Similarly we can prove a(bac)a=a(abc) a= a(cba)a =a(bca)a= a(acb)a

Therefore (T,.) is normal

Case (ii): If b ρ a ⇔ b^3 = bab = aba

b(abc)b = (bab)cb

= b^3cb

= bcb

= bcb^3

= bc(bab)

b(abc)b = b(cba)b

Similarly we can prove a(bac)a=a(abc) a= a(cba)a =a(bca)a= a(acb)a

Therefore (T,.) is normal.

Theorem 3.12: If (T,., ρ) is a totally ordered ternary semigroup such that relation 'ρ' on T defined by a ρ b ⇔ a^3 = aba=bab for all a,b in T.If (T,.) is R-commutative then (T,.) is right regular.

Proof: Given that (T,., ρ) is a totally ordered ternary semigroup which is semilattice and (T,.) is R-commutative .i.e. abc = xy(bac) a, b, c in T and x,y in T^1.

We have to prove (T,.) is right regular for this consider the following cases

Case (i): If a ρ b ⇔ a^3 =aba=bab

xya^3 = xy (aba)

= b a a

= aba

= a^3

xya^3 = a

Therefore (T,.) is right regular

Case (ii): b ρ a ⇔ b^3 =bab=aba

xyb^3 = xy(bab)

= a b b

= b a b

= b^3

xyb^3 = b

Therefore (T,.) is right regular

Theorem 3.13: Suppose (T,., ρ) is a totally ordered ternary semigroup where ' ρ' on T defined as a ρ b ⇔ a^3 =aba=bab for all a,b in T. If (T,.) is L-commutative then (T,.) is left regular.

Proof: As given (T,., ρ) is a totally ordered ternary semigroup which is semilattice and (T,.) is L-commutative .i.e. abc = (acb)xy for all a , b, c in T and x,y in T^1.

To prove that (T,.) is left regular consider the following cases

Case (i): If a ρ b ⇔ a^3 =aba=bab

a^3xy = (aba)xy

= a a b

= aba

= a^3

a^3xy = a

Therefore (T,.) is left regular.

Case (ii) : If b ρ a ⇔ b^3 =bab=aba

b^3xy = (bab)xy

= b b a

= bab

= b^3

b^3xy = b

Therefore (T,.) is left regular.

Definition 3.14: A ternary semigroup (T,.) is called anti regular if ababa= b for all a,b in T

Theorem 3.15: Assume that (T,., ρ) is a totally ordered ternary semigroup such that relation ' ρ' on T is given by a ρ b ⇔ a^3 =aba=bab for all a,b in T .If (T,.) is lateral singular then it is antiregular.

Proof: Given that (T,., ρ) is a totally ordered ternary semigroup which is band and (T,.) is lateral singular. We have to prove (T,.) is antiregular for this consider the following cases

Case (i): If a ρ b ⇔ a^3 =aba=bab

ababa = (aba)ba

= a^3ba

= aba

ababa = b (since (T,.) is lateral singular)

Therefore (T,.) is antiregular

Case (ii): If b ρ a ⇔ b^3 =bab=aba

babab = b^3ab

= bab

babab = a (since (T,.) is lateral singular)

Therefore (T,.) is antiregular.

Conclusion: In this paper we mainly studied about totally ordered ternary semigroup with properties like R-Commutative, L-Commutative, Intra-regular, quasi-commutative, Anti-regular, normal etc

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