Solution of Cubic Equation of Stress Invariants for a Particular Condition

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Abstract - *Analysis of stresses for an infinitesimal tetrahedron leads to a situation where we obtain a cubic equation consisting of three stress invariants i.e. linear invariant, quadratic invariant and cubic invariant. The solution of such cubic equation, which yields the principal stresses, cannot be done by any direct method and therefore is solved by trial and error method or by any suitable numerical technique. The trial and error methods being time-consuming numerical techniques are much preferred approach for obtaining solutions. In the present study an approach has been made to obtain solution for such equations for a particular case. The approach gives direct solution to the problem and therefore one does not need to resort to cumbersome trial and error methods or the iterative numerical procedure.*

Key Words: **Cubic Equation, Stress Invariant, Trigonometric, Explicit, Principal Stress**

1. INTRODUCTION

Analysis of an infinitesimal tetrahedron leads to a cubic equation consisting of three stress invariants, which are linear, quadratic and cubic. Past researchers [1] have indicated that the solution of such equation cannot be done by any direct method and one has to carry out trial and error method or numerical method consisting of several iterations. These methods are time consuming and cumbersome. Researchers, however, had derived some explicit method for solution of cubic equations consisting deviatoric invariants [1]. In this present study an attempt has been made to obtain solutions for the cubic equation for a particular case and determine the principal stresses.

2. DESCRIPTION

Let us consider an infinitesimal tetrahedron OABC having an oblique plane ABC, which intersects Cartesian X, Y & Z axes at A, B & C respectively (Figure 1a). The stresses on various planes are indicated in the Figure 1a. The plane normal ON on ABC passes through the center of the axes O. The direction cosines of ABC are (l,m,n) where $l = \cos\alpha_1, m = \cos\alpha_2, n = \cos\alpha_3$ (Figure 1b).

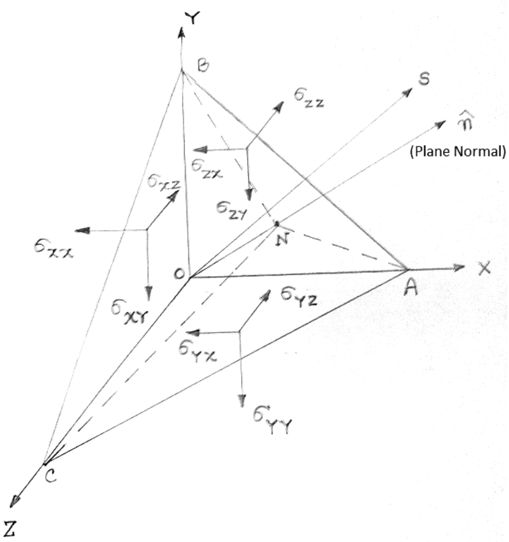

Figure –1a: Stresses on an Infinitesimal Tetrahedron

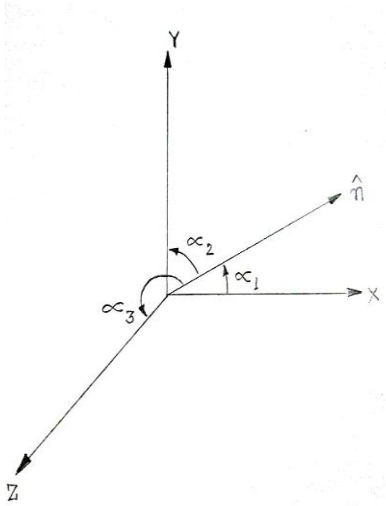

Figure –1b: Directional angles of Plane Normal

Assuming $S_x, S_y \& S_z$ the three stress components along the 3 Cartesian directions on the oblique plane and considering equilibrium by applying Newton's second law of motion in the X, Y & Z directions we get,

$$\begin{Bmatrix} S_x \\ S_y \\ S_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}.$$

This 3x3 stress matrix related to the equilibrium of deformed body is a symmetric matrix and is known as Eulerian Stress Matrix, which is sometimes called as Cauchy Stress Matrix. The resultant stress is given as, $S = \sqrt{S_x^2 + S_y^2 + S_z^2}$ and the normal stress is given as $S_n = S_x \times l + S_x \times m + S_x \times n$. Now, if the resultant stress coincides with the normal stress i.e. the direction magnitudes of both become same the shear stress on the oblique plane becomes zero and the same plane becomes the principal plane resulting S_x, S_y & S_z to be equal to $S \times l, S \times m$ & $S \times n$ respectively. Therefore,

$$S \times l = \sigma_{xx} \times l + \sigma_{yx} \times m + \sigma_{zx} \times n$$
$$S \times m = \sigma_{xy} \times l + \sigma_{yy} \times m + \sigma_{zy} \times n$$
$$S \times n = \sigma_{xz} \times l + \sigma_{yz} \times m + \sigma_{zz} \times n$$

and from that we obtain

$$\begin{bmatrix} \sigma_{xx} - S & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - S & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - S \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = 0.$$

Since $l = m = n = 0$ is not possible as $l^2 + m^2 + n^2 = 1$, we can clearly say that,

$$\begin{vmatrix} \sigma_{xx} - S & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - S & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - S \end{vmatrix} = 0.$$

Expanding the determinant we can obtain,

$$S^3 - I_1 \times S^2 - I_2 \times S - I_3 = 0$$

$-$ Cubic Equation of Invariants (A)

where,

$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ (Linear Invariant)

$I_2 = \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 - \sigma_{xx} \times \sigma_{yy} - \sigma_{yy} \times \sigma_{zz} - \sigma_{zz} \times \sigma_{xx}$

(Quadratic Invariant)

$I_3 = 2 \times \sigma_{xy} \times \sigma_{yz} \times \sigma_{zx} + \sigma_{xx} \times \sigma_{yy} \times \sigma_{zz} -$
$\sigma_{xx} \times \sigma_{yz}^2 - \sigma_{yy} \times \sigma_{xz}^2 - \sigma_{zz} \times \sigma_{xy}^2$

(Cubic Invariant)

The solution of this equation can be done by trial and error method, which is an extremely time consuming and cumbersome method. Therefore, the roots of this equation are usually determined by numerical technique and the Newton-Raphson method is preferred for that as the method gives the roots with a sufficiently high accuracy.

If we now consider a particular situation where,

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$$

i.e. the linear invariant is zero ($I_1 = 0$) the cubic equation becomes,

$$S^3 - I_2 \times S - I_3 = 0.$$

If we put $S = r \times \sin\theta$ into the cubic equation we get,

$$\sin^3\theta - \left(\frac{I_2}{r^2}\right) \times \sin\theta - \left(\frac{I_3}{r^3}\right) = 0.$$

Observing similarity between this reduced equation and the trigonometric identity - $\sin^3\theta - \left(\frac{3}{4}\right) \times \sin\theta + \left(\frac{1}{4}\right) \times \sin 3\theta = 0$ (obtained from $\sin 3\theta = 3 \times \sin\theta - 4 \times \sin^3\theta$) we obtain

$$r = \left(\frac{2}{\sqrt{3}}\right) \times \sqrt{I_2} \quad \text{and}$$

$$\sin 3\theta = -\left(4 \times \frac{I_3}{r^3}\right) = -\left(\left(\frac{3\sqrt{3}}{2}\right) \times \left(\frac{I_3}{I_2\sqrt{I_2}}\right)\right) = k.$$

The above yields three possible values of $\sin\theta$ and subsequently three principal stresses σ_1, σ_2 & σ_3 -

$$\theta_1 = \left(\frac{1}{3}\right) \times \sin^{-1}(k) + \left(\frac{4\pi}{3}\right) = \theta_1 - \left(\frac{2\pi}{3}\right)$$
$$\theta_2 = \left(\frac{1}{3}\right) \times \sin^{-1}(k)$$
$$\theta_3 = \left(\frac{1}{3}\right) \times \sin^{-1}(k) + \left(\frac{2\pi}{3}\right) = \theta_1 + \left(\frac{2\pi}{3}\right).$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \left(\frac{2}{\sqrt{3}}\right) \times \sqrt{I_2} \times \begin{Bmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \sin\theta_3 \end{Bmatrix}$$

Therefore, this gives the solution of the cubic equation for a particular condition i.e. $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$ explicitly and gives the principal stresses without any trial and error or any iterative procedure.

3. EXAMPLE

A solution of an example problem [3] is given here to illustrate the concept. The stress matrix for an infinitesimal body is given as,

$$[S] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}.$$

The invariants for this can be calculated as,

$$I_1 = 0,\ I_2 = 7\ \&\ I_3 = -6.$$

The cubic equation therefore becomes,

$$S^3 - 7 \times S + 6 = 0.$$

Now considering the similarity, as has been explained, we obtain,

$$r = 2 \times \sqrt{7/3} = \sqrt{(4 \times 7)/3} = \sqrt{28/3} \text{ and}$$

$$\sin 3\theta = -\frac{4 \times (-6)}{(\sqrt{28/3})^3} = 0.8417.$$

From this we get, $\theta_1 = 259.1,\ \theta_2 = 19.1\ \&\ \theta_3 = 139.1$ and

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \left(\frac{2}{\sqrt{3}}\right) \times \sqrt{I_2} \times \begin{Bmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \sin\theta_3 \end{Bmatrix} = \left(\frac{2}{\sqrt{3}}\right) \times \sqrt{7} \times \begin{Bmatrix} \sin 259.1 \\ \sin 19.1 \\ \sin 139.1 \end{Bmatrix} = \begin{Bmatrix} -3 \\ 1 \\ 2 \end{Bmatrix}.$$

In this way we can find out the principal stresses without any trial and error or any iterative method for such problems where the linear invariant is zero.

4. CONCLUSION

The aim of this study was to find out a procedure for the explicit solution for the cubic equation of stress invariants. On such purpose an attempt has been made to solve the cubic equation considering the similarity between the cubic equation of invariants and the trigonometric equation of $\sin\theta$ for a particular condition that is when the linear invariant is zero. It has been found that the approach gives accurate explicit solution in terms of principal stresses and excludes the need for any trial and error method or any iterative method.

5. REFERENCES

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BIOGRAPHIES

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