ISOLATE DOMINATION IN QUASI-TOTAL GRAPHS

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Abstract - *A dominating set S of a graph G is said to be an isolate dominating set of G if induced subgraph on S has at least one isolated vertex. An isolate dominating set S is said to be a minimal isolate dominating set if no proper subset of S is an isolate dominating set. The isolate domination number γ_0 is defined as the minimum cardinality of an isolate dominating set. In this paper, we investigate the isolate domination number of quasi-total graph of certain classes of graphs.*

***Key Words*: Dominating set, Domination number, Isolate dominating set, Isolate domination number, Quasi- Total graph.**

1. INTRODUCTION

By a graph G, we mean a finite, undirected graph with neither loops nor multiple edges. In a graph $G = \big(V(G), E(G)\big)$, $V(G)$ is the vertex set and $E(G)$ is the edge set of G. The degree of a vertex v is the number of edges of G incident with v and is denoted by $deg_G(v)$ or $deg\, v$. For vertices u and v in a connected graph G, the distance $d(u,v)$ is the length of a shortest $u-v$ path in G. The eccentricity $e(u)$ of a vertex u in G is defined by $e(u) = \max\{d(u,v): v \in V(G)\}$. The maximum eccentricity among the vertices of G is its diameter which is denoted by $diam(G)$. A vertex v in a connected graph G is called a peripheral vertex if $e(v) = diam(G)$. The sub graph induced by a set X of vertices of a graph G is denoted by $\langle X\rangle$ with $V(\langle X\rangle) = X$ and $E(\langle X\rangle) = \{uv \in E(G) : u, v \in X\}$. Let u and v be vertices of a graph G. A $u-v$ walk of G is a finite alternating sequence $u = u_0, e_1, u_1, e_2, \dots\dots,$ $e_n, u_n = v$ of vertices and edges in G beginning with a vertex u and ending with a vertex v such that $e_i = u_{i-1}u_i; i = 1,2, \dots, n$. The number n is called the length of the walk. The walk is said to be open if u and v are distinct vertices; it is closed otherwise. A walk in which all the vertices are distinct is called a path. A closed walk $u_0, u_1, u_2, \dots\dots, u_n$ in which $u_0, u_1, u_2, \dots\dots, u_{n-1}$ are distinct is called a cycle. A set D of vertices of a graph G is said to be a dominating set if every vertex in $V - D$ is adjacent to a vertex in D. A dominating set D is said to be a minimal dominating set if no proper subset of D is a dominating set. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$ [2]. A dominating set S of graph G is said to be an isolate dominating set if $\langle S\rangle$ has at least one isolated vertex [2]. An isolate dominating set S is said to be a minimal isolate dominating set if no proper subset of S is an isolate dominating set [2]. The minimum cardinality of an isolate dominating set is called isolate domination number $\gamma_0(G)$ for a graph G. The quasi-total graph P(G) of a graph G is the graph whose vertex set is V(G) ∪ E(G) and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of G or two adjacent edges of G or one is a vertex and other is an edge incident with it in G. This concept was introduced in [3].

Domination becomes the most fascinating topic in graph theory due to its application in networking. The study of isolate domination was initiated by Sahul Hamid in 2013 [1]. The study of isolate domination motivated us to introduce isolate domination in quasi-total graphs. Any undefined term or notation in this paper can be found in Harary [4].

2. Main Results

In this section, we study isolate domination number γ_0 of quasi-total graphs of complete graphs, complete bipartite graphs, wheels, cycles and paths.

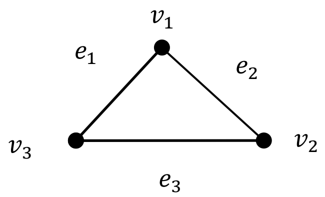
Theorem 2.1. $\gamma_0\big(P(K_n)\big) = \left\lceil \frac{n}{2} \right\rceil$ for $n \geq 1$.

Proof. Let $v_1, v_2, \dots\dots, v_n$ be the vertices and $e_1, e_2, \dots\dots, e_{\binom{n}{2}}$ be the edges of K_n. Then $P(K_n)$ has the vertices $v_1, v_2, \dots\dots, v_n, e_1, e_2, \dots\dots, e_{\binom{n}{2}}$. Hence $|V(P(K_n))| = n + \binom{n}{2}$. Since the edge $e_i\ \left[1 \le i \le \binom{n}{2}\right]$ of K_n is adjacent to $2(n-2)$ edges and incident with 2 vertices say, v_a, v_b $(1 \le a \ne b \le n)$ of K_n, the vertex $e_i\ \left[1 \le i \le \binom{n}{2}\right]$ of $P(K_n)$ is adjacent to $2n-2$ vertices of $P(K_n)$. Hence $e_i\ \left[1 \le i \le \binom{n}{2}\right]$ dominates $2n-1$ vertices including itself in $P(K_n)$.

Therefore, there are $\binom{n-1}{2}$ vertices in $P(K_n)$ which need to be dominated. Let such v_j's $(1 \le j \ne a \ne b \le n)$ and e_k's $\left(1 \le k \ne i \le \binom{n}{2}\right)$ belong to $E = \left\{v_1, v_2, \dots\dots, v_{n-2}, e_1, e_2, \dots\dots, e_{\binom{n-2}{2}}\right\}$. Consider a vertex e_1 belongs to E in $P(K_n)$. Let $e_1 = \{v_p, v_q\}$ in K_n where $1 \le p \ne q \ne a \ne b \le n$. Since v_p and v_q are adjacent to all the other vertices in K_n, e_1 is adjacent to $2n-4$ vertices belonging to E in $P(K_n)$. Hence e_1 dominates $2n-3$ vertices belonging to E in $P(K_n)$ including itself.

Of the remaining vertices that are not dominated, Consider a vertex e_2. It is adjacent to $2n-6$ vertices after excluding vertices to e_1 and already dominated vertices. Hence e_2 dominates $2n-5$ vertices including itself. This process continues till $2n-(2k-1) \ge n+1$, where $k = 1, 2, \dots\dots, \frac{n}{2}$ whenever K_n is of even n. Also $2n-(2k-1) \ge n$, where $k = 1, 2, \dots\dots, \frac{n+1}{2}$ whenever K_n is of odd n.

Therefore, $\gamma_0(P(K_n)) = \left\lceil \frac{n}{2} \right\rceil$.



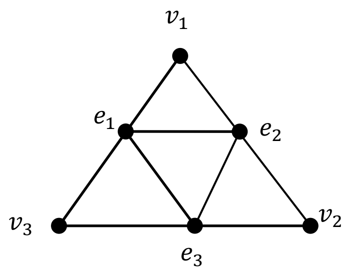


Figure 1: K_3 and $P(K_3)$

Example 2.2. Consider K_3 with vertex set $V = \{v_1, v_2, v_3\}$ and edge set $E = \{e_1, e_2, e_3\}$. $P(K_3)$ has vertex set as $V(P(K_3)) = \{e_1, e_2, e_3, v_1, v_2, v_3\}$ and $|V(P(K_3))| = 6$. e_1 will dominate $\{e_1, e_2, e_3, v_1, v_3\}$ and v_2 will be dominated by itself. Therefore, $S = \{e_1, v_2\}$. $|S| = 2$. Hence, $\gamma_0(P(K_3)) = \left\lceil \frac{3}{2} \right\rceil = 2$.

Theorem 2.3. $\gamma_0\left(P(K_{m,n})\right) = \begin{cases} n \ if \ m = n \\ n+1 \ if \ m > n \ge 1 \end{cases}$

Proof. Let $V_1 = \{v_1, v_2, \dots\dots, v_m\}$ and $V_2 = \{v_1', v_2', \dots\dots, v_n'\}$ be the two bipartites of $V(K_{m,n})$. Let $e_1, e_2, \dots\dots, e_{mn}$ be the edges of $K_{m,n}$. Hence $\left|V\left(P(K_{m,n})\right)\right| = m + n + mn$. Since an edge $e_1$of $K_{m,n}$ is incident with 2 vertices and adjacent to $m-1+n-1$ edges of $K_{m,n}$, the vertex e_1 of $P(K_{m,n})$ is adjacent to $m+n$ vertices of $P(K_{m,n})$. Hence e_1 dominates $m+n+1$ vertices including itself in $P(K_{m,n})$.

Therefore, there are $mn-1$ vertices in $P(K_{m,n})$ need to be dominated. Let $e_1 = \{v_a, v_b'\}$ in $K_{m,n}$ where $1 \le a \le m$ and $1 \le b \le n$. Consider a vertex e_2 of $P(K_{m,n})$ which is not dominated by e_1. Let $e_2 = (v_c, v_d')$ where $1 \le a \ne c \le m$ and $1 \le b \ne d \le n$. Clearly e_2 is incident with two vertices and $m+n-2$ edges of $K_{m,n}$. Since the vertices v_a, v_d' and v_c, v_b' are adjacent in $K_{m,n}$, e_2 dominates $m+n-2$ vertices of $P(K_{m,n})$ after excluding vertices to e_1 . Hence e_2 dominates $m+n-1$ vertices of $P(K_{m,n})$ including itself.

Therefore, there are $mn-(m+n-1)-1$ vertices of $P(K_{m,n})$ which need to be dominated. Consider a vertex e_3 of $P(K_{m,n})$ which is not dominated by e_1 and e_2. Clearly, e_3 dominates $m+n-3$ vertices of $P(K_{m,n})$ including itself. Therefore, there are $mn-2(m+n-2)-1$ vertices of $P(K_{m,n})$ which need to be dominated. Continuing like this, we finally get a vertex e_n of $P(K_{m,n})$ dominates $m-n-(n-1)$ vertices of $P(K_{m,n})$ including itself. Therefore, there are $m-n$ vertices of $P(K_{m,n})$ which need to be dominated.

If $m=n$, then all the vertices of $P(K_{m,n})$ are dominated by the set $S=\{e_1,e_2,\ldots\ldots,e_n\}$. Hence, $\gamma_0\left(P(K_{m,n})\right)=n$.

If $m>n$, then $m-n$ vertices are remaining to dominate. Clearly, these $m-n$ vertices are adjacent with each other. Hence, $\gamma_0\left(P(K_{m,n})\right)=n+1$.

Therefore, $\gamma_0\left(P(K_{m,n})\right)=\begin{cases} n \text{ if } m=n \\ n+1 \text{ if } m>n\geq 1\end{cases}$

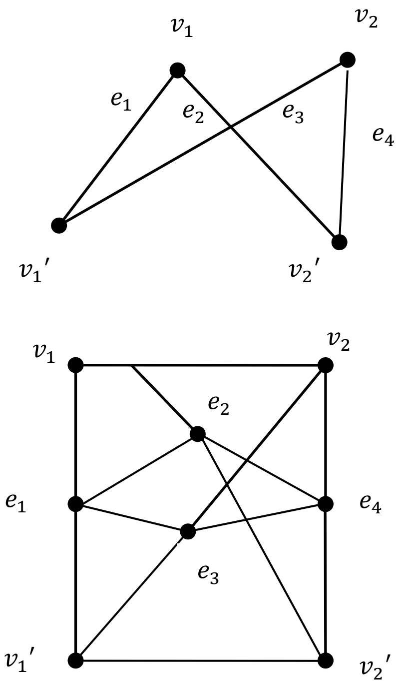


Figure 2: $K_{2,2}$ and $P(K_{2,2})$

Example 2.4. Consider $K_{2,2}$ with vertex set $V=\{v_1,v_2,v_1{}',v_2{}'\}$ and edge set $E=\{e_1,e_2,e_3,e_4\}$. Then $P(K_{2,2})$ has vertex set as $V\left(P(K_{2,2})\right)=\{e_1,e_2,e_3,e_4,v_1,v_2,v_1{}',v_2{}'\}$. Here e_1 will dominate $\{e_1,e_2,e_3,v_1,v_1{}'\}$ and e_4 will dominate $\{e_2,e_3,e_4,v_2,v_2{}'\}$. Therefore, $S=\{e_1,e_4\}$. $|S|=2$. Hence $\gamma_0\left(P(K_{2,2})\right)=2$.

Theorem 2.5. $\gamma_0(P(W_n))=\left\lceil\frac{n-5}{3}\right\rceil+3$ for $n\geq 4$.

Proof. Let $v_1,v_2,\ldots\ldots,v_{n+1}$ be the vertices of the wheel W_n, where $v_1$is the centre vertex of W_n. Let $e_1,e_2,\ldots\ldots,e_{2n}$ be the edges of W_n. Let $e_1,e_2,\ldots\ldots,e_n$ be the edges with end vertices as peripheral vertices and $e_{n+1},e_{n+2},\ldots\ldots,e_{2n}$ the edges with v_1 as one end vertex and other end vertex to be $v_2,v_3,\ldots\ldots,v_{n+1}$ respectively. Hence total number of vertices in $P(W_n)$ is $|V(P(W_n))|=3n+1$. Let e_i $(n+1\leq i\leq 2n)$ be one of the edges with v_1 as a one end vertex and other end vertex to be one of $v_2,v_3,\ldots\ldots,v_{n+1}$ of W_n. Clearly e_i $(n+1\leq i\leq 2n)$ is adjacent to all the other e_j's $(n+1\leq i\neq j\leq 2n)$ and exactly two vertices and two edges which are adjacent to the peripheral vertices in W_n. Hence e_i $(n+1\leq i\leq 2n)$ is adjacent to $n+3$ vertices in $P(W_n)$. Hence e_i $(n+1\leq i\leq 2n)$ dominates $n+4$ vertices including itself in $P(W_n)$.

Therefore, there are $2n-3$ vertices need to be dominated in $P(W_n)$. Among these $2n-3$ vertices, $n-2$ are edges and $n-1$ are vertices in W_n. Consider an end vertex among these $n-1$ vertices of W_n which need to be dominated in $P(W_n)$. Clearly it is adjacent to remaining $n-3$ vertices and one edge of W_n in $P(W_n)$ which are not dominated before. Therefore, there are $n-3$ edges and a single vertex of W_n which are need to dominate in $P(W_n)$. Consider an edge e_i of W_n which is adjacent to a remaining vertex v_j of W_n in $P(W_n)$. Clearly e_i is adjacent to another edge e_j of W_n which is not already dominated in $P(W_n)$. Hence there are $n-5$ edges of W_n which are need to dominate in $P(W_n)$. Let such vertices of $P(W_n)$ be $e_1,e_2,\ldots\ldots,e_{n-5}$

Clearly, each e_i $(1\leq i\leq n-5)$ is adjacent to two e_j's incident on its vertices in W_n. Hence e_i dominates 3 vertices including itself in $P(W_n)$. Therefore, $\left\lceil\frac{n-5}{3}\right\rceil$ number of vertices require to dominate remaining $n-5$ vertices in $P(W_n)$.

Therefore, $\gamma_0(P(W_n)) = \left\lceil\frac{n-5}{3}\right\rceil + 3$

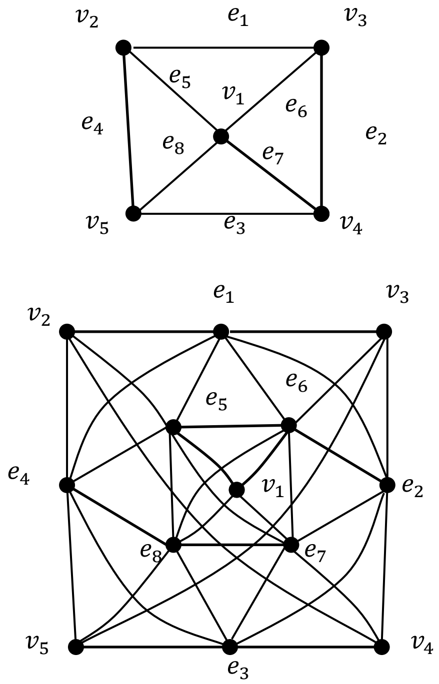


Figure 3: W_4 and $P(W_4)$

Example 2.6. Consider W_4 with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$. Then $V(P(W_4)) = \{v_1, v_2, v_3, v_4, v_5, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$. Here e_5 dominates $\{v_1, v_2, e_1, e_4, e_5, e_6, e_7, e_8\}$, e_3 dominates $\{v_4, v_5, e_2, e_3, e_4, e_7, e_8\}$ and remaining v_3 is dominated by v_3. Therefore, $S = \{e_3, e_5, v_3\}$ and $|S| = 3$. Hence $\gamma_0(P(W_4)) = \left\lceil\frac{4-5}{3}\right\rceil + 3 = 3$.

Theorem 2.7. $\gamma_0(P(C_n)) = \left\lceil\frac{n-3}{3}\right\rceil + 2$ for $n \geq 5$.

Proof. Let $v_1, v_2, \ldots\ldots, v_n$ be the vertices and $e_1, e_2, \ldots\ldots, e_n$ be the edges of C_n. Then $P(C_n)$ has the vertices $v_1, v_2, \ldots\ldots, v_n, e_1, e_2, \ldots\ldots, e_n$. Hence $|V(P(C_n))| = 2n$. Clearly each v_i $(1 \leq i \leq n)$ of C_n is adjacent to two vertices and two edges of C_n. Hence each v_i $(1 \leq i \leq n)$ is adjacent to $n-1$ vertices of $P(C_n)$. Hence v_i $(1 \leq i \leq n)$ dominates n vertices including itself in $P(C_n)$.

Therefore there are n vertices in $P(C_n)$ which need to be dominated. Among these n vertices, $n-2$ are edges and two are vertices in C_n. Consider a vertex among these 2 vertices of C_n which need to be dominated in $P(C_n)$. Clearly it is adjacent to remaining one vertex and one edge of C_n in $P(C_n)$ which are not dominated before. Therefore, there are $n-3$ vertices of $P(C_n)$ which are need to dominate. Let such vertices of $P(C_n)$ be $e_1, e_2, \ldots\ldots, e_{n-3}$.

Clearly, each e_i $(1 \leq i \leq n-3)$ is adjacent to two e_j's incident on its vertices in C_n. Hence e_i dominates 3 vertices including itself in $P(C_n)$. Therefore, $\left\lceil\frac{n-3}{3}\right\rceil$ number of vertices require to dominate remaining $n-3$ vertices in $P(C_n)$.

Therefore, $\gamma_0(P(C_n)) = \left\lceil\frac{n-3}{3}\right\rceil + 2$

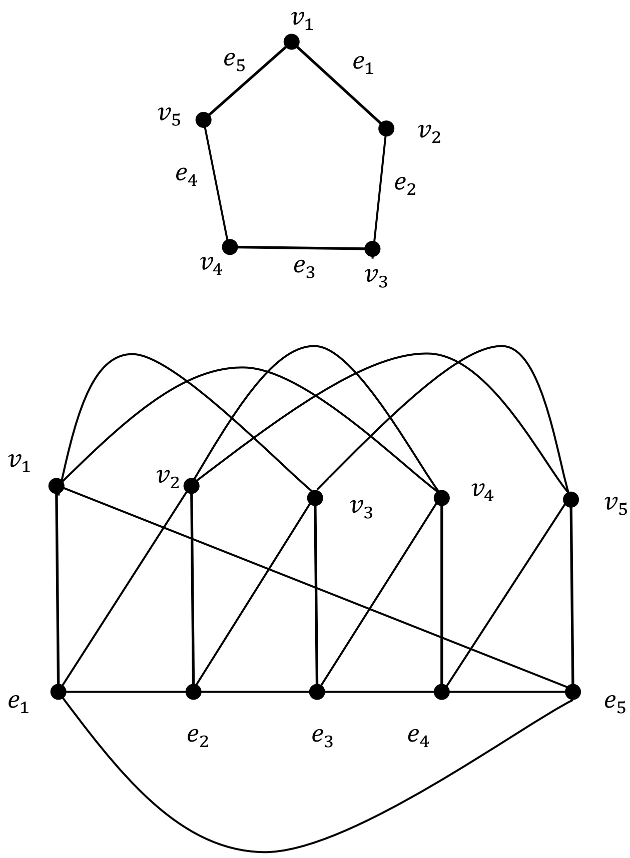


Figure 4: C_5 and $P(C_5)$

Example 2.8. Consider C_5 with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5\}$. Then $V(P(C_5)) = \{e_1, e_2, e_3, e_4, e_5, v_1, v_2, v_3, v_4, v_5\}$. Here e_1 dominates $\{e_1, e_2, e_5, v_1, v_2\}$, e_3 dominates

$\{v_3, v_4, e_2, e_3, e_4\}$ and remaining v_5 is dominated by v_5. Therefore, $S = \{e_1, e_3, v_5\}$ and $|S| = 3$. Hence $\gamma_0(P(C_5)) = \left\lceil \frac{5-3}{3} \right\rceil + 2 = 3$.

Remark 2.9. $\gamma_0(P(C_3)) = 2$ and $\gamma_0(P(C_4)) = 2$

Theorem 2.10. $\gamma_0(P(P_n)) = 2 + \left\lceil \frac{n-4}{3} \right\rceil$ for $n \geq 3$.

Proof. Let $v_1, v_2, \ldots\ldots, v_n$ be the vertices and $e_1, e_2, \ldots\ldots, e_{n-1}$ be the edges of P_n. Then $P(P_n)$ has the vertices $v_1, v_2, \ldots\ldots, v_n, e_1, e_2, \ldots\ldots, e_{n-1}$. Hence $|V(P(P_n))| = 2n - 1$. Clearly each v_i $(i = 1, n)$ of P_n is adjacent to $(n-2)v_j$'s and an edge of P_n in $P(P_n)$. Hence v_i $(i = 1, n)$ dominates n vertices including itself in $P(P_n)$.

Therefore, there are $n - 1$ vertices in $P(P_n)$ which need to be dominated. Among these $n - 1$ vertices, $n - 2$ are edges and one is a vertex in P_n. Consider an edge e_j of P_n adjacent to v_j where $i \neq j$. Clearly e_j dominates v_j and one more edge of P_n which is not already dominated. Hence e_j dominates 3 vertices including itself in $P(P_n)$.

Therefore, there are $n - 4$ vertices in $P(P_n)$ which need to be dominated. Since each e_i has two e_j's incident on its vertices in P_n, e_i dominates 3 vertices including itself in $P(P_n)$. Therefore, $\left\lceil \frac{n-4}{3} \right\rceil$ number of vertices require to dominate remaining vertices in $P(P_n)$.

Hence, $\gamma_0(P(P_n)) = 2 + \left\lceil \frac{n-4}{3} \right\rceil$.

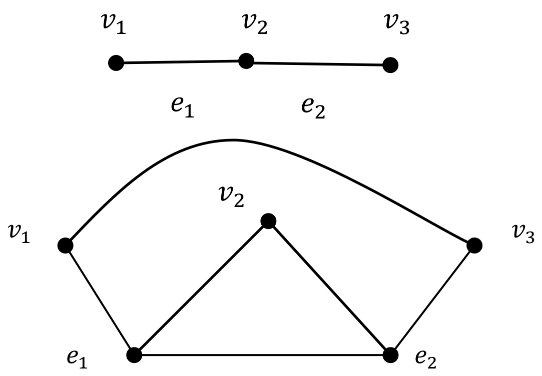


Figure 5: P_3 and $P(P_3)$

Example 2.11. Consider P_3 with vertex set $V = \{v_1, v_2, v_3\}$ and edge set $E = \{e_1, e_2,\}$. Then $V(P(P_3)) = \{e_1, e_2, e_3, v_1, v_2, v_3\}$. Here e_1 dominates $\{e_1, e_2, v_1, v_2\}$ and the remaining v_3 is dominated by v_3. Therefore, $S = \{e_1, v_3\}$ and $|S| = 2$. Hence $\gamma_0(P(C_5)) = 2 + \left\lceil \frac{3-4}{3} \right\rceil = 2$.

Remark 2.12. $\gamma_0(P(P_2)) = 1$

3. CONCLUSION

In this paper, we have discussed isolate domination in line graphs of complete graphs, complete bipartite graphs, wheels, cycles and paths. In future, our focus will be on doing study of isolate domination in middle graphs.

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