Wavelet Approximation Order in Different Smoothness Spaces by Composition of Functions

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Abstract

In this paper, our aim is to approximate a given composition of function by a finite term from the wavelet series. The approximation order is improved if the order of smoothness of the given function is improved, discussed in previous literature like by DeVore (1998), Cohen (2003) and Siddiqi (2004). But we observe that in the case of nonlinear approximation, the approximation order is improved quicker than that in linear case. In this paper we satisfy this assumption only for the Haar wavelet.

Introduction

Approximation using superposition of functions has existed since the early 1800's when Joseph Fourier discovered that he could superpose sine and cosine to represent other functions. Wavelets are functions that satisfy certain mathematical requirements and are used in representing data or other functions. Wavelet algorithms process data at different scales or resolutions. This makes wavelet interesting and useful. Wavelet are well suited for approximating data with sharp discontinuities.

The fundamental idea behind wavelets is to analyze according to scale. The wavelet analysis procedure is to adopt a wavelet prototype function called an analyzing wavelet or mother wavelet. A resent development of approximation theory is approximation of an arbitrary function by wavelet polynomials. There are different types of wavelet such as Haar wavelet, Maxican-Hat wavelet, Shannon wavelet, Daubechies wavelet, Meyer's wavelet. In this paper we mainly focus on approximation by dilated Haar wavelet.

In wavelet theory, if we approximate the target function by selecting terms of the wavelet series, for which the target function f is kept controlled only over the number of terms to be used, it is called N-term approximation. Our purpose is to approximate a function via dilated Haar wavelet in section 2 we show a brief discussion on Haar wavelet and its properties. In section 3 we approximate a function by dilated Haar wavelet in different smoothness space.

Infinite series is a mathematical tool for exact representation of certain functions. When we work with the series expansion representation we are only able to deal with finite sums. For example, if a function f has an exact representation through Fourier series, we need to have finite partial sum $(S_N)_{N\epsilon N}$ for computer well. For a good approximation N becomes very large. In section 4 we use only few Haar coefficients for which it is nonlinear. In that case we get a significant improvement of approximation order in comparison to any other wavelet method.

Dilated Haar System

Definition 2.1 *(Haar function)- The Haar Mother wavelet*

$$\Psi(x) = \varphi(2x) - \varphi(2x-1) = f(x) = \begin{cases} 1 \; if \; x \in \left[0, \frac{1}{2}\right) \\ -1 \; if \; x \in \left[\frac{1}{2}, 1\right) \\ 0 \; otherwise \end{cases}$$

The Mother wavelet Ψ generates

$$\Psi_{j,k}(x) = 2^{\frac{j}{2}} \Psi\left(2^j t - k\right), j \in N \cup \{0\}, k = 0,1, \ldots . 2^{j-1}. (1)$$

Here j is the generation of $\Psi_{j,k}$. The collection Ψ of the $\Psi_{j,k}$ together with φ is called the set of Haar functions.

Definition 2.2 *(characteristic function) – For any set $E \subset R$, define the function $\chi_E : R \rightarrow R$ by*

$$\chi_E(x) = \begin{cases} 1 \; if \; x \in E \\ 0 \; if \; x \notin E \end{cases}$$

We call χ_E the characteristic function of E.

Definition 2.3 *(dyadic interval) – Define the interval $I_{j,k}$ by $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$ which is known as dyadic interval. For each pair of $j, k \in Z$. The collection of all such intervals is called dyadic subintervals of R.*

The function $g(x) = \varphi(bx)$ is called a dilation of function f by b. If we dilate $\varphi(x)$ by b we get the characteristic function of the interval $\left(0, \frac{1}{b}\right)$ that is $(bx) = 1_{\left[0, \frac{1}{b}\right]}(x)$.

Now we take dilation factor 2 *. we get* $\varphi(2x) = 1_{\left[0,\frac{1}{2}\right)}(2x)$.

Now we take $j = 1$ *and* $x = 2x$ *in equation* (1) *we get*

$$\Psi_{1,0}(2x) = 2^{\frac{1}{2}}\Psi(4x) \text{ and } \Psi(2x) = \begin{cases} \sqrt{2} \text{ if } x \in \left[0,\frac{1}{8}\right) \\ -\sqrt{2} \text{ if } x \in \left[\frac{1}{8},\frac{1}{2}\right) \\ 0 \text{ elswhere} \end{cases} \text{ is called dilated Haar wavelet.}$$

Definition 2.4 *–(Haar Scaling function) – The set of expansion functions composed of integer translations and unary scaling of real, square integrable function* $\varphi(x)$*, that is the set* $\{\phi_{j,k}(x)\}$ *where* $\phi_{j,k} = 2^{\frac{j}{2}}\phi(2^j x - k)$ *for all* $j, k \in Z$ *and* $\varphi(x) \in L^2(R)$*.* $\Phi(x)$ *is called scaling function.*

Consider $f(x) = L^2(R)$ *relative to wavelet* $\Psi(x)$ *and scaling function* (x) *. We can write*

$$f(x) = \textstyle\sum_k C_{j_0}(k)\Phi_{J_0,K}(x) + \sum_{j=j_0}^{\infty}\sum_k^{\infty} d_j(k)\Psi_{j,k}(x).$$

Here $C_{j_0}(k)$ *is called approximation or scaling coefficients and* $d_j(k)$ *is wavelet coefficients.*

$$C_{j_0}(k) = \langle f(x), \Phi_{j_0,k}(x)\rangle = \int f(x)\Phi_{j_0,k}(x)dx$$

And $d_j(k) = \langle f(x), \Psi_{j,k}(x)\rangle = \int f(x)\Psi_{j,k}(x)dx$

Now we consider $f(x)$ *is defined on* $L_2[0,1]$ *, for any integer* $j \geq 0$.

$$f(x) = \sum_{k=0}^{2^j-1}\langle f, \varphi_{j,k}\rangle\Phi_{j,k}(x) + \sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1}\langle f, \Psi_{j,k}\rangle\Psi_{j,k}(x)$$

$$f(x) = \textstyle\sum_{k=0}^{2^j-1} C_{j,k}\Phi_{j,k}(t) + \sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\Psi_{j,k}(t)$$

Which is known as Haar series and $d_{j,k}$ *and* $c_{j,k}$ *are the Haar coefficient for wavelet and Haar scaling coefficient, respectively.*

Approximation by dilated Haar wavelet in different spaces

Approximation space:

Let $(X, ||.||_X)$ be a Normed space in which the approximation takes place. Let $(S_N)_{N\geq 0}$ be a family of subspace of a normed space X. Our approximation comes from the space $(S_N)_{N\geq 0}$ is a subset of $X \subset R$

For a function $f \in X$, the approximation error is

$$E_N(f)_X = dist.(f, S_N)_X = inf_{g \in S_N} \left\{ \left\| f - g \right\|_X \right\},$$

where g is the approximation function in $(S_N)_{N\geq 0}$

For linear approximation. N represent the number of parameters, which are needed to describe an element in S_N. That is, N is dimention of S_N. In most cases of interest $E_N(f)$ goes to zero as N tends to infinity.

For nonlinear approximation: N is related to the number of free parameter. For example, N might be the number of knots in piecewise content approximation with free knots, The S_N can be quite general space , in particular ,they do not have to be linear.

Approximation in $L_2(R)$.

Let f be a function on $L_2(R)$ and g is also continuous all those points where f is discontinuous such that fog is also continuous on $L_2(R)$ and the Haar wavelet series of h is

$$h = gof \approx \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \langle fog, \Psi_{j,k} \rangle \Psi_{j,k}(t)$$

if $\Psi_{j,k}(t)$ is support on the interval $I_{j,k} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]$, then

$$\langle gof, \Psi_{j,k}(t) \rangle = \int_R gof(t) \Psi_{j,k}(t)\ dt$$

$$\langle gof, \Psi_{j,k}(t) \rangle = \int_R g(f(t)) \Psi_{j,k}(t)\ dt$$

$$\langle gof, \Psi_{j,k}(t) \rangle = \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} g(f(t))\, 2^{-\frac{j}{2}} \Psi(2^j t - k)\ dt$$

$$\langle gof, \Psi_{j,k}(t) \rangle = 2^{-\frac{j}{2}} \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} g(f(t))\, \Psi(2^j t - k)\, dt \quad (2)$$

For computing finite sum, Let $N = 2^j$ *be coefficients for some* $\in N$ *. We consider* $j = 0,1,2,3 \ldots..j-1$ *then*

$$\sum_{j=0}^{j-1} \sum_{n=0}^{2^j-1} 1 + 2 + 2^2 + \cdots + 2^j - 1 = N - 1$$

cofficients. For Haar wavelet we can see that for each j only one of the coefficients is nonzero and it is size is $2^{\frac{-j}{2}}$ *, for detail one see Christensen and Christensen [2] and Walnut [7].*

Then the error of the approximation in $L_2(R)$ *is*

$$||gof - \sum_{j=J}^{j-1} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x)||_{L_2}^2 = ||\sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x)||_{L_2}^2$$

$$||gof - \sum_{j=J}^{j-1} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x)||_{L_2}^2 = \left(\sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \left| \langle gof, \Psi_{j,k} \rangle \right|^2 \right)$$

$$||gof - \sum_{j=J}^{j-1} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x)||_{L_2}^2 \approx \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |\langle gof, \Psi_{j,k} \rangle|^2$$

$$||gof - \sum_{j=J}^{j-1} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x)||_{L_2}^2 \approx \sum_{j=J}^{\infty} \left(2^{\frac{-j}{2}}\right)^2 \approx 2^{-j}$$

$$\approx \frac{1}{N} = O(2^{-j})$$

Approximation in $L_p(R)$*.*

THEOREM 1: I*f* $f, g \in L_p(R)$ *and the partial sum of the Haar wavelet series of* gof *is the Haar wavelet series of f and g is* $h = \sum_{j=J}^{j-1} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x)$ *, for* $\in N$ *, then the error of the approximation is* $O(2^{-j})$*.*

PROOF. The error of the approximation in $L_p(R)$ *is*

$$||gof - h||_{L_p} = \left\| gof - \sum_{j=J}^{j-1} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x) \right\|_{L_P}$$

$$||gof - h||_{L_p} = \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x) \right\|_{L_P}$$

$$||gof - h||_{L_p} = \left(\sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |\langle gof, \Psi_{j,k} \rangle|^P \right)^{\frac{1}{P}}$$

$$||gof - h||_{L_p} \approx \left(\sum_{j=J}^{\infty} 2^{\frac{-jp}{2}} \right)^{\frac{1}{P}} \approx 2^{\frac{-j}{2}} = O\left(2^{\frac{-J}{2}}\right) \quad (3)$$

Approximation in $Lip_M(\alpha, L_p)$ Space

THEOREM 2 If $f, g \in Lip_M(\alpha, L_p)$, $0 < \alpha \leq 1, 1 < p \leq \infty, M > 0$, *and*

$h(t) = \sum_{j=J}^{j-1} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x)$ *is the Haar wavelet series of gof f for some* $j \in N$, *then the error of the approximation in* $Lip_M(\alpha, L_p)$ *is* $O(2^{-J\alpha})$.

PROOF: From DeVore [4] we have If $f, g \in Lip_M(\alpha, L_p)$, $0 < \alpha \leq 1, 1 < p \leq \infty$,

$$dist(f, S_N)_p \leq inf_{g \in S_N} ||gof - h||_p \leq C_p |f|_{lip(\alpha, L_p)} \delta^{\alpha},$$

where $\delta = max_{0 \leq k < N} |t_{k+1} - t_k|$.

So the error of approximation in in $Lip_M(\alpha, L_p)$ *is*

$$||gof - h||_{L_p} = \left\| gof - \sum_{j=J}^{j-1} \sum_{k=0}^{2^j-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x) \right\|_{L_p}$$

$$||gof - h||_{L_p} \leq C_p |f|_{lip(\alpha, L_p)} (2^{-J})^{\alpha}$$

$$||gof-h||_{L_p} \leq M(2^{-J})^{\alpha} = O(2^{-J\alpha}) \ (4)$$

Where C_p is depending on p.

Approximation in sobolve space $H^m(R)$.

THEOREM 3 *If $f, g \in H^m(R)$ and $h(x) = \sum_{j=J}^{j-1}\sum_{k=0}^{\infty}\langle gof, \Psi_{j,k}\rangle \Psi_{j,k}(x)$ is the finite Haar wavelet series of gof for some $J \in N$, then the error of the approximation is $O(2^{\frac{-mN}{2}})$, where $N = 2^J$.*

Proof : The error of the approximation is

$$||gof-h||_{L_2} = \left\|\sum_{j=J}^{\infty}\sum_{k=0}^{N-1}\langle gof, \Psi_{j,k}\rangle \Psi_{j,k}(x)\right\|_{L_2}$$

$$||gof-h||_{L_2} \leq \left(\sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1}|\langle gof, \Psi_{j,k}\rangle|^2\right)^{\frac{1}{2}}$$

$$||gof-h||_{L_2} \leq \left(\sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1}\frac{2^{mk}}{2^{mN}}|\langle gof, \Psi_{j,k}\rangle|^2\right)^{\frac{1}{2}}$$

$$||gof-h||_{L_2} \leq 2^{-mN}\left(\sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1}2^{mk}|\langle gof, \Psi_{j,k}\rangle|^2\right)^{\frac{1}{2}}$$

By using the properties of Besove space we have

$$||f||_{H^m(L_2(R))} \cong \left(\left(\sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1}2^{mk}|\langle gof, \Psi_{j,k}\rangle|^2\right)^{\frac{1}{2}}\right) \ (5)$$

Therefore

$$||gof-h||_{L_2} \leq 2^{\frac{-mN}{2}}||f||_{H^m(L_2(R))} = O(2^{\frac{-mN}{2}}).$$

Approximation in Besove Space $B_q^{\alpha}(L_p(R))$.

THEOREM 4 *If $f, g \in B_q^{\alpha}(L_q(R))$, $\alpha > 0, 0 < q \leq \infty$, and*

$$h(x) = \sum_{j=0}^{j-1}\sum_{k=0}^{N}\langle gof, \Psi_{j,k}\rangle \Psi_{j,k}(x)$$

Is the finite Haar wavelet series of gof for some j ∈ N, then the approximation is $O\left(2^{\frac{-\alpha N}{2}}\right)$.

Where $N = 2^J$.

Proof The error of the approximation is

$$||gof - h||_{L_q} = \left\|\sum_{j=J}^{\infty}\sum_{k=0}^{N-1}\langle gof, \Psi_{j,k}\rangle \Psi_{j,k}(x)\right\|_{L_q}$$

$$||gof - h||_{L_q} \leq \left(\sum_{j=J}^{\infty}\sum_{k=o}^{N-1}\left|\langle gof, \Psi_{j,k}\rangle\right|^q\right)^{\frac{1}{q}}$$

$$||gof - h||_{L_q} \leq \left(\sum_{j=J}^{\infty}\sum_{k=o}^{N-1}\frac{2^{\alpha k}}{2^{\alpha N}}\left|\langle gof, \Psi_{j,k}\rangle\right|^q\right)^{\frac{1}{q}}$$

$$||gof - h||_{L_q} \leq 2^{\frac{-\alpha N}{q}}\left(\sum_{j=J}^{\infty}\sum_{k=o}^{N-1}2^{\alpha k}\left|\langle gof, \Psi_{j,k}\rangle\right|^q\right)^{\frac{1}{q}}$$

By using the properties of Besove space we have

$$||f||_{B_q^m(L_q(R))} \cong \left(\sum_{j=J}^{\infty}\sum_{k=o}^{N-1}2^{\alpha k}\left|\langle gof, \Psi_{j,k}\rangle\right|^q\right)^{\frac{1}{q}} \quad (6)$$

Therefor

$$||gof - h||_{L_{q(R)}} \leq 2^{\frac{-\alpha N}{q}}\, ||f||_{B_q^{\alpha}(L_q(R))} = O\left(2^{\frac{-\alpha N}{q}}\right)$$

where $\frac{1}{q} = \frac{\alpha}{2} + \frac{1}{2}$.

CONCLUSION

The above theorem shows that the approximation order will improve if the smoothness of the approximation space is improved.

Nonlinear Approximation by Haar wavelet

Our previous discussion is finite linear approximation by Haar wavelet. Now we consider nonlinear approximation via Haar wavelet. We have see that for level j ,exactly one Haar coefficient is nonzero. One can see Christensen and Christensen [2] and Walnut[7].

If we can calculate $N = 2^J$ biggest Haar coefficient, in that case the approximation error is

$$\sigma_N(f)_X = dist.(f, S_N)_X = inf_{g \in \sum N} ||f - g||_X \ldots\ldots\ldots\ldots.(7)$$

Where $\sum N$ and $\sigma_N(f)$ denote the set of wavelet and approximation error, respectively, in the nonlinear space .

Nonlinear approximation in $L_p(R)$

THEOREM 4.1 If $f, g \in L_p(R)$ and the partial sum of the Haar wavelet series of gof is the Haar wavelet series of f and g is $h = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x)$, for $j \in N$, then the error of the nonlinear approximation is $O(2^{-\frac{N}{p}})$.

Proof: The error of the nonlinear approximation in $L_p(R)$ is

$$||gof - h||_{L_p} = \left\| gof - \sum_{j=0}^{N-1} \sum_{k=0}^{2^N-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x) \right\|_{L_P}$$

$$||gof - h||_{L_p} = \left\| \sum_{j=0}^{\infty} \sum_{k=0}^{2^N-1} \langle gof, \Psi_{j,k} \rangle \Psi_{j,k}(x) \right\|_{L_P}$$

$$||gof - h||_{L_p} = \left(\sum_{j=0}^{\infty} \sum_{k=0}^{N-1} |\langle gof, \Psi_{j,k} \rangle|^P \right)^{\frac{1}{P}}$$

$$||gof - h||_{L_p} \approx \left(\sum_{j=N+1}^{\infty} 2^{\frac{-Np}{2}} \right)^{\frac{1}{P}} \approx 2^{\frac{-N}{2}} = O\left(2^{\frac{-N}{2}}\right) \ldots\ldots\ldots\ldots\ldots.(8)$$

CONCLUSION

From the above discussion we have seen that in case of nonlinear approximation the approximation order depends on the order of smoothness of the function space. But in the case of nonlinear approximation there is a significant improvement in the approximation order compared to that in linear approximation. Also if *we assume that $g(x) = I(x)$ identity function and then we get all results of M. R. ISLAM et al. [8]. That is our paper is proper generalization and extension of results of M. R. ISLAM et al. [8].*

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