On Some Minimal S-Quasinormal Subgroups of Finite Groups

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Abstract: *A subgroup H of a group G is permutable subgroup of G if for all subgroups S of G the following condition holds SH = HS < S,H >. A subgroup H is S-quasinormal in G if it permutes with every Sylow subgroup of G. In this article we study the influence of S-quasinormality of subgroups of some subgroups of G on the super- solvability of G.*

I.INTRODUCTION:

When H and K are two subgroups of a group G, then HK is also a subgroup of G if and only if HK = KH. In such a case we say that H and K permute. Furthermore, H is a permutable subgroup of G, or H permutable in G, if H permutes with every subgroup of G. Permutable subgroups where first studied by Ore [7] in 1939, who called them quasinormal. While it is clear that a normal subgroup is permutable, Ore proved that a permutable subgroup of a finite group is sub- normal. We say, following Kegel [6], that a subgroup of G is S-quasinormal in G if it permutes with every Sylow subgroup of G. Several authors have investigated the structure of a finite group when some subgroups of prime power order of the group are well-situated in the group. Buckley [2] proved that if all maximal subgroups of an odd order group are normal, then the group is super solvable. It turns out that the group which has many S-quasinormal subgroups have well-described structure.

II .PRELIMINARIES

CONJECTURE 1.1.

If H_i is a permutable subgroup of G for all i ∈ I , then

$< H_i : i \in I >$ *is a permutable subgroup of G*.

CONJECTURE 1.2.

Let H *and* K *be subgroups of* G *such that* K ≤ H *and* K ✂G*. Then* H *is a permutable subgroup of* G *if and only if* H/K *is a permutable subgroup of* G/K.

CONJECTURE 1.3.

If H *is a permutable subgroup of* G *and* S *is a subgroup of* G *,then* H ∩ S *is a permutable subgroup of* S.

CONJECTURE 1.4.

Let H *be a* p*-subgroup of* G *for some prime* p*. Then*

$H \in Syl(G)^{\perp}$ *if and only if* $N_G(H) = O^p(G)$.

III.MAIN RESULT

THEOREM 2.1 *Let* p *be the smallest prime dividing* |G|. *If* P *is a Sylow* p*-subgroup of* G *such that every minimal subgroup of* P *is* S-*quasinormal in* G*, then* G *has a normal* p*-complement.*

Proof;

Let H be a minimal subgroup of P . It follows from 1.4 that $N_G(H)$ contains $O^p(G)$. Since $P \le N_G(H)$ we have that H is normal in G. Suppose that P has at least two distinct minimal subgroups H_1 and H_2. Then $H_1 H_2 = P$. Hence P is normal in G. Let r be a prime different from p and R Be a Sylow r-subgroup of G. By the above and 1.4 R normalizes each minimal Subgroup of P .

Since p is a smallest prime dividing |G|, we have that R induces a trivial automorphism group on P/Φ(P) (Φ(P) is a Frattini subgroup of P). This implies G = P ⊠ T by Schur Theorem.

Now we may assume that P has only one minimal subgroup H. Then P is cyclic and the assertion follows from Burnside's transfer theorem.

REMARK. It follows from 1.4 that if a minimal subgroup of a Sylow p-subgroup of a group G is S-quasinormal, then it is also normal in G. Moreover G is even p-decomposable, if its Sylow p-subgroup for smallest prime p is non-cyclic and every minimal subgroup of its Sylow p-subgroup is S-quasinormal.

COROLLARY 2.2 *. Put* π (G) = $\{p_1, p_2, \ldots, p_n\}$ *. Let* P_i *be a Sylow* p_i*-subgroup of* G*, where* i = 1, 2, . . . , n*. If every minimal subgroup of* P_i *is* S*-quasinormal in* G *for all* i ∈ {1, 2, . . . , n}*, then* G *is supersolvable*

Proof.

Let $p_1 > p_2 >$ ⊠⊠⊠> p_n. By Theorem 2.1 G has a normal p_n-complement K. If a Sylow p_n-subgroup P_n is non-cyclic, then by Remark we have G = K ⊠P_n. By induction, K is supersolvable. Therefore, G is supersolvable too. Suppose that P_n is cyclic. Then G = K oP_n, a semidirect product of a normal subgroup K and P_n. By induction K is supersolvable. Moreover all non-cyclic Sylow subgroups of K are normal in G. Denote by H the direct product of all non-cyclic Sylow subgroups of G. Clearly, H is a nilpotent normal Hall subgroup of G. The Frattini subgroup Φ(H) is normal in G and the group G/Φ(H) by 1.2 satisfies the condition of the corollary. By induction we may assume that G/Φ(H) is a supersolvable group provided Φ(H) = 1.

Since the formation U of all supersolvable groups is saturated, this implies that G is supersolvable. Hence we may assume that Φ(H) = 1. we have that H is a direct product of elementary abelian p_i-subgroups for all $p_i \in \pi(H)$. By Schur-Zassenhaus theorem on existence of complements we have G = H o L where L is a Hall subgroup of G with cyclic Sylow p-subgroups for all p ∈ π(L). Now it is enough to show that P o L is a supersolvable group for each Sylow p-subgroup of H. But every minimal subgroup of P is normal in G (see Remark) and the result follows.

THEOREM 2.3. *If a group* G *has a normal* p*-subgroup* P *such that* G/P *is supersolvable and every minimal subgroup of* P *is* S*-quasinormal in* G*, then* G *is supersolvable*.

Proof.

We prove the theorem by induction on |G|. Let P_1 be a Sylow p-subgroup of G. If P = P_1, then by Remark after Theorem 2.1 we have G = P_1 o R where R is a Hall p^0-subgroup of G, isomorphic to G/P . It is easy to see that the Frattini subgroup Φ(P) is in the Frattini subgroup of G. If Φ(G) is non-trivial, then G/Φ(G) is supersolvable by 1.2 and induction. Since the formation U of all supersolvable groups is saturated this implies the supersolvability of G. Hence we may assume that Φ(P) = 1. P is an elementary abelian group. Now the result follows from Remark after Theorem 2.1 If P = P_1 is cyclic, then G is clearly supersolvable.

Suppose that P < P_1. We may assume that P is non-cyclic. Since G is solvable, it has a Hall p^0-subgroup H. By Remark after Theorem 2.1 it follows that the subgroup K = HP = H ⊠ P. Clearly P is normal in P_1. Hence Z(P_1) ∩ P is non-trivial. Let Z be a cyclic subgroup of order p in P ∩ Z(P_1). Since G = P_1H, we have Z is normal in G. By induction and 1.2 we get G/Z is supersolvable. Now we obtain the required assertion from the definition of supersolvable group.

COROLLARY 2.4.

Let N *be a normal subgroup of* G *such that* G N *is supersolvable and* π(N) = $\{p_1, p_2, \ldots, p_s\}$*. Let* P_i *be a Sylow* p_i*-subgroup of N, where* i = 1, 2, . . . , s*. Suppose that all minimal subgroups of each* P_i *are* S*-quasinormal in* G*. Then* G *is supersolvable*.

Proof.

We prove the theorem by induction on |G|. From Corollary 2.2 we have N has an ordered Sylow tower. Hence if p_1 is the largest prime in π(N), then P_1 is normal in N. Clearly, P_1 is normal in G. Observe that (G P_1) (N P_1) ~= G N is supersolvable. Therefore we conclude that G P_1 is supersolvable by induction on |G|. Now it follows from Theorem 2.3 that G is supersolvable

THEOREM 2.5.

Let P *be a Sylow* p-*subgroup of* G *where* p *is the smallest prime dividing* |G|. *Suppose that all minimal subgroups of* Ω(P) *are* S-*quasinormal in* G. *Then* G *has a normal* p-*complement.*

Proof.

Let H be a minimal subgroup of Ω(P) .Our hypothesis implies that H is S-quasinormal in G and so O^p (G) ≤ N_G (H) ≤ G by 1.4. Clearly, HO^p (G) ≤ N_G (H) ≤ G . If HO^p (G) ≤ N_G (H) < G, then HO^p (G) has a normal p- complement K by induction. Thus K is a normal Hall p^0 -subgroup of G and so G has a normal p-complement. Now we may assume that N_G (H) = G, i.e. H is normal in G. If G has no normal p-complement, then by Frobenius theorem, there exists a nontrivial p-subgroup L of G such that N_G (L) /C_G (L) is not a p-group. Clearly we can assume that L ≤ P . Let r be any prime dividing |N_G (L)| with r = p and let R be a Sylow r-subgroup of N_G (L). Then R normalizes L and so Ω (L) R is a subgroup of N_G (L). Since H is normal in G, we have HΩ(L)R is a subgroup of G. Now Theorem 2.`implies that (HΩ (L)) R has a normal p-complement and so also Ω (L) R. Since Ω (L) R has a normal p-complement, R, and Ω (L) is normalized by R, then Ω (L) R = Ω (L) � R and so by [5, Satz 5.12, p. 437], R centralized L. Thus for each prime r dividing |N_G (L)| with r = p, each Sylow r-subgroup R of N_G (L) centralized L and hence N_G (L) /C_G (L) is a p-group; a contradiction. Therefore G has a normal p-complement.

COROLLARY 2.6. *Put* π (G) = {p_1, p_2, . . . , p_n} *where* p_1 > p_2 > ���> p_n. *Let* P_i *be a Sylow* p_i-*subgroup of* G *where* i = 1,2,. . .,n. *Suppose that all minimal subgroups of* Ω (P_i) *are* S-*quasinormal in* G. *Then* G *possesses an ordered Sylow tower*.

LEMMA 2.7. *Suppose that* P *is a normal* p-*subgroup of* G *such that* G P *is supersolvable. Suppose that all minimal subgroups of* Ω (P) *are* S-*quasinormal in* G. *Then* G *is supersolvable.*

Proof.

We prove the lemma by induction on |G|. Let P_1 be a Sylow p-subgroup of G. We treat the following two cases:

Case 1. P = P_1. Then by Schur- Zassenhous theorem, G possesses a Hall p� subgroup K which is a complement to P in G. The G P ~= K is supersolvable. Since Ω (P) char P and P is normal in G, it follows that Ω (P) is normal in G. Then Ω (P) K is a subgroup of G. If Ω (P) K = G, then G Ω (P) is supersolvable. Therefore G is supersolvable by Theorem 2.3 Thus we may assume that Ω (P) K < G. Since Ω (P) K Ω (P) ~= K is supersolvable, it follows by Theorem 2.3 that Ω (P) K is supersolvable. we conclude the supersolvability of G.

Case 2. P < P_1. Put π (G) = {p_1, p_2, . . . , p_n}, where p_1 > p_2 > ���> p_n. Since G P is supersolvable, it follows by [1] that G P possesses supersolvable subgroups H P and K P such that |G P : H P | = p_1 and |G P : K P | = p_n. Since H P and K P are supersolvable, it follows that H and K are supersolvable by induction on |G|. Since |G : H| = |G P : H P | = p_1 and |G : K| =|G P : K P | = p_n, it follows again by [1] that G is supersolvable.

THEOREM 2.8. *Put* π (G) = {p_1, p_2, . . . , p_n} *where* p_1 > p_2 > ���> p_n. *Let* P_i *be a Sylow* p_i-*subgroup of* G *where* i = 1, 2, . . . , n. *Suppose that all minimal subgroups of* Ω (P_i) *are* S-*quasinormal in* G. *Then* G *is supersolvable.*

Proof.

We prove the theorem by induction on |G|. By Theorem 2.5 and Lemma 2.7 we have that G possesses an ordered Sylow tower. Then P_1 is normal in G. By Schur-Zassenhaus' theorem, G possesses a Hall p�subgroup K Which is complement to P_1 in G. Hence K is supersolvable by induction. Now it follows from Lemma 2.7 that G is supersolvable.

COROLLARY 2.9

Let N *be a normal subgroup of* G *such that* G N *is supersolvable. Put* π (N) = {p_1, p_2, . . . , p_s}, *where* p_1 > p_2 > ���> p_s. *Let* P_i *be a Sylow* p_i- *subgroup of* N*. Suppose that all minimal subgroups of* Ω (P_i) *are* S-*quasinormal in* N*. Then* G *is supersolvable.*

Proof.

We prove the corollary by induction |G|. Theorem 2.8 implies that N is supersolvable and so P_1 is normal in N, where P_1 is Sylow p_1-subgroup of N and p_1 is the largest prime dividing the order of N. Clearly, P_1 is normal in G. Since (G P_1) (N P_1) ~= G N is supersolvable, it follows that G P_1 is supersolvable by induction on |G|. Therefore G is supersolvable by Lemma 2.7 The corollary is proved.

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