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TRIPLE FACTORIZATION OF NON-ABELIAN GROUPS BY TWO MINIMAL SUBGROUPS

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Abstract. *The triple factorization of a group G has been studied recently showing that G = ABAfor some proper subgroups A and B of G, the definition of rank-two geometry and rank-two coset geometry which is closely related to the triple factorization was de fined and calculated for abelian groups. In this paper we study two infinite classes of non-abelian finite groups D2n and P SL (2, 2n) for their triple factorizations by finding certain suitable minimal subgroups, which these subgroups are define with original generators of these groups. The related rank-two coset geometries motivateus to define the rank-two coset geometry graphs which could be of intrinsic tool on the study of triple factorization of non-abelian groups.*

Keywords: Rank-two geometry, triple factorization, dihedral groups, projective special linear groups.

I. Introduction

The factorization of a finite group G as the inner product G = ABA where, A and B are proper subgroups of G, the notation T = (G, A, B) is used for a triple factorization of the group G. finite simple groups and their automorphism groups were studied .The aim of this paper is to study the rank-two coset geometry by defining a graph, which is named a rank-two coset geometry graph. The notation Γ(G, A, B) will be used for this graph, where G = ABA. Our computational results based on the study of two classes of non-abelian groups D2n (the dihedral group of order 2n) and the projective special linear groups P SL(2, 2n), (n ≥ 3). The nice and very interesting presentation of projective special linear groups may be found in ([5, 6, 7]) and the related references.

It is necessary to recall that for studying the triple factorization of groups the important tools come from permutation group theory and we recall some of them which will be useful in our proofs. The set of all permutations of a set Ω is the symmetric group on Ω, denoted by Sym(Ω), and a subgroup of Sym(Ω) is called a permutation group on Ω. If a group G acts on Ω we denote the induced permutation group of G by GΩ, a subgroup of Sym(Ω). We say that G is transitive on Ω if for all α, β ∈ Ω there exists g ∈ G such that αg = β. For a transitive group G on the set Ω, a nonempty subset Δ of Ω is called a block for G if for each g ∈ G, either Δg = Δ, or Δg ∩ Δ = ∅; in this case the set Σ = {Δg|g ∈ G} is said to be a block system for G. The group G induces a transitive permutation group GΣ on Σ, and the set stabi- lizer GΔ induces a transitive permutation group GΔ on Δ. If the only blocks for G are the singleton subsets or the whole of Ω we say that G is primitive, and otherwise G is imprimitive.

II. PRELIMINARIES

Definition 2.1. A triple factorization T = (G, A, B) of a finite group G is called degenerate if G = AB or G = BA. Otherwise, T = (G, A, B) is called a non-degenerate triple factorization. A group with a triple factorization T = (G, A, B), is sometimes called an ABA-group.

Definition 2.2. Let P and L be the sets of right cosets of the proper subgroups A and B of a finite group G, respectively. The property ∗ between the elements of P and L which is named a ”non-empty inter- section relation” is defined as follows:

Ax ∗ By ⇐⇒ Ax ∩ By = ∅

Then (Ω = P ∪ L, ∗) is called a rank-two coset geometry and will be denoted by Cos(G, A, B).

In a rank-two coset geometry, if the property ∗ holds between two members Ax ∈ P and By ∈ L, then we say that these members are incident, and in this case the pair (Ax, By) is called a flag of rank-two coset geometry.

Definition 2.3. The rank-two coset geometry graph of a finite non- abelian group G will be denoted by Γ(G, A, B), is an undirected graph with the vertex set P ∪ L and two points Ax and By are adjacent if and only if Ax ∩ By = ∅ where, G = ABA.

III.MAIN RESULT

Theorem 3.1. Let G = D_{2n} = ha, b|a^n = b^2 = $(ab)^2$ = 1i be the dihedral group of order 2n. Then,

(1) For n = 3k, (k = 1, 2, ...), there are at least two proper dihedral subgroups B and C of G such that G=BCB (non-degenerate triple factorization).

(2) For n = 2^k,, (k = 1, 2, ...), there is no non-degenerate triple factorization for G.

(3) For the prime values of n ≥ 5, there is no non-degenerate triple factorization for G.

(4) The graph associated to a triple factorization T = (G, A, B) of G, (Γ(G, A, B)) is bipartite graph if and only if the factorization is degenerate.

Proof:

(1)Forn=3k,(k=1,2,3,...),D_{2n}=ha,b|a^{3k}=b^2=$(ab)^2$=1i anditsdihedralsubgroupsareintheform<a^d,a^ib>where, d≥3,d|n=3kand0≤i≤d−1.NowifB=<a^r,a^ib> andC=<a^s,a^jb>betwodistinctdihedralsubgroupofD$_{2n}$=$D_{2(3k)}$ such that |B||C||B| ≥ 2n, thenforsomei,j,l,m,nthat,0≤i,j,l≤n−1and0≤m,n≤1,thereexistelementsx=a^ib^m∈B,y=a^jb^n∈Candg=a^l∈D_{2n}suchthat By=Bgx.So,byLetT=(D_{2n},A,B)isatriple factorizationofD_{2n},andbyusingtherelationsba^ib=a^{-i}, a^iba^{-i}=a^{2i}b,(a^ib)b$(a^ib)^{-1}$=a^{2i}banda^{i}ba^{-i}=b,(0≤i≤ n−1)ofD_{2n}wegetthatforevery0≤r,s,l≤n−1and0≤α,β,γ≤1,theworda^r$b^\alpha$$a^s$$b^\beta$$a^l$$b^\gamma$ofBCBisoneofthe elementsofD_{2n}.So,thistriplefactorizationisnon-degenerate andD_{2n}=BCB=CBC.(2) Forn=2^k,(k=1,2,3,...),byLemma2.1,thenumberofnon-trivialcyclicanddihedralsubgroupsofD_{2n}iskand2^{k+1}−2,respectively.Inthecasek=1,thenon-trivialcyclicsubgroupof D_4isA=<a>={1,a}andthenontrivialdihedralsubgroups areB=<a^2,a^0b>=<1,b>={1,b}andC=<a^2,a^1b>=< 1,ab>={1,ab},suchthatbyusingtherelationsofD_{2n}we get,AB=BA=AC=CA=BC=CB=D_4.And foreveryk≥2,itiseasytoseethatforthecyclicsubgroup A=<a^1>andforanytwodistinctnontrivialdihedralsub-groupsBandCsatisfyingB*C,C*Band|B||C||B|≥2n wegetAB=BA=AC=CA=BC=CB=D_{2n}.Hence,the triples(D_{2n},A,B),(D_{2n},A,C)and(D_{2n},B,C)aredegenerate triplefactorizations.

(3)Fortheprimevaluesofn≥5,thenumberofnontrivialcyclic anddihedralsubgroupsofD_{2n}are1andn,respectively,where A=<a>istheonlynontrivialcyclicsubgroupandforevery i(i=0,1,...,n−1),B_i=<a^n,a^ib>isanontrivialdihedral subgroup.ByusingtherelationsofD_{2n}onemayseethatfor every1≤i,j≤n−1,AB_iA=AB_i=D_{2n}but$B_iB_jB_i$=D_{2n}. Thus,inthiscasethereisnonon-degeneratetriplefactorization forD_{2n}.

(4)By(2)and(3),T=(D_{2n},A,B_i)isadegeneratetriplefactor-izationof$D_{2n}$where,A=<a>istheonlycyclicsubgroup ofD_{2n}ofindex2andB_i=<a^n,a^ib>,(i=0,1,...,n−1)is adihedralsubgroupofindexn,wheren≥5isaprimeand thesetofdistinctrightcosetsofAand$B_i$are{A,Ab}and{B_i,B_ia,B_ia^2,...,B_ia^{n-1}},respectively.Byusingtherelations ofD_{2n}wegetthatforevery0≤i,k≤n−1,A∩$B_ia^k$and Ab∩B_ia^karenotempty.Sobythedefinitionofrank-twocoset geometry,foreveryi,(i=0,1,...,n−1),eachcosetofAis adjacenttoallcosetsofB_i.Therefore,Γ(D_{2n},A,B_i)=$K_{2,n-1}$, thecompletebipartitegraph.Bythesamemethodonemayseethatif$T=(D_{2n},A,B)$isadegeneratetriplefactorization fortwodistinctsubgroupsAandB,then$\Gamma(D_{2n},A,B)=K_{r,s}$ where,randsaretheindicesofthesubgroupsAandB,respectively.Fortheinversecase,let$\Gamma(D_{2n},B,C)=K_{p,q}$.the bydefinitionofrank-twogeometrygraphpandqaretheordersoftwodistinctpropersubgroups$B=<a^r,a^ib>$and $C=<a^s,a^jb>$,where$|D_{2n}:B|=r,|D_{2n}:C|=s$and thesetofrightcosetsofBandCare$\{B,Ba,Ba^2,...,Ba^{r-1}\}$ and$\{C,Ca,Ca^2,...,Ca^{s-1}\}$,respectively.Nowbyconsidering theelementsofsubgroupsB,CandD_{2n}onemayseethat $D_{2n}=BC$andthetriplefactorizationisdegenerate. ✷

Lemma 3.2.

EverysubgroupofD$_{2n}$(n≥3),iscyclicoradihedral groupsuchthat:

(i)thecyclicsubgroupsare$<a^d>$,whered|nand$|D_{2n}:<a^d>|=2d$,

(ii)thedihedralsubgroupsare$<a^d,a^ib>$,whered|n,and0≤i≤d−1,and$|D_{2n}:<a^d,a^ib>|=d$

(iii) let n be odd and m|2n. For odd values of m there are m sub- groups of index m inD_{2n} However, if m is even there is exactly one subgroup of index m,

(iv) let n be even and m|2n. For odd values of m there are m sub- groups of index m. If m is even and doesn't divide n, there is only one subgroup of index m. Finally, if m is even and m|n, there are exactly m + 1 subgroups of index m.

There are also certain obvious relations in D_{2n} Indeed, for every integer i = 1, 2, ..., n, the following relations hold in D_{2n}:

$ba^ib=a^{-i},a^iba^{-i}=a^{2i}b,(a^ib)b(a^ib)^{-1}=a^{2i}b,a^iba^{-i}=b$

Lemma 3.3.

Let A and B be two proper subgroups of a group G, and consider the right coset action of G on $\Omega_A = \{Ag|g \in G\}$. Set $\alpha = A \in\Omega_A$. Then T = (G, A, B) is a triple factorization if and only if the B-orbit α^BintersectsnontriviallyeachG_α-orbitinΩ_A.

Lemma 3.4 Let A and B be two proper subgroups of a group G and consider the right coset action of G on $\Omega_A = \{Ag|g \in G\}$. Set $\alpha = A \in\Omega_A$ Then, T = (G, A, B) is a triple factorization if and only if for all g ∈ G there exists elements b ∈ B, a ∈ A such that Ab = Aga.

Lemma 3.5. For any two proper and distinct subgroups A and B of if $T = (D_{2n}, A, B)$ is a degenerate (non-degenerate) triple factorization for D2n then $T = (D_{2n}, B, A)$ is also a degenerate (non- degenerate) triple factorization for D_{2n}. Moreover, D_{2n} = ABA = BAB.

Proof. The proof is easy by using Lemma 2.3 and the relations of D_{2n}

Lemma 3.6. There are exactly $P_2(n)$presentations for the group

P SL(2,2^n), (n ≥ 3), where $P_2(n)(n)$ = 1 d|n μ(n)2d and μ is the Mobius

Proof. In the relation $x^n=yxy^{a_{n-1}}xy^{a_{n-2}}...xy^{a_0}$of Sinkov's presentation, every choice of $a_0,a_1,...,a_{n-1}$yields an irreducible polynomial over GF (2) of degree n. On the other hand by the elementary results of [12], the number of such polynomials is $P_2(n)(n)$ = 1 P d|n μ(n)2d, where μ is the Mobius function. where, for at least a primitive α of GF (2^n)), m(α) = 0. So, the number of distinct presentations for P SL(2, 2^n)) is $P_2(n)$.

Lemma 3.7. For every integer n ≥ 3, the last relation of the pre- sentation of P SL(2,2^n) will be reduced to x^n=yx^{n-1}yxyorx^n= $yx^{n-2}yx^2y$, if n is even either is odd.

Proof. For n = 3, $P_2(3)$ = 2 (the number of irreducible polynomials of degree 3 over GF (2)) and one of these polynomials is the trinomial m(x) =x^3+x^2+1. For n = 4, $P_2(4)$ = 3 and one of these polynomials is the trinomial m(x) = x^4+x+1.. On the other hand by using the results of [14] we deduce that, for every integer n ≥ 3, at least one of the irreducible polynomials of degree n is a trinomial, and this trinomial is in the form m(x) = x^n+x^2+1or m(x) = x^n+ x + 1 when n is odd either n is even, respectively. Now, by considering the coefficients of this trinomials we see that the relation $x^n=yxy^{a_{n-1}}xy^{a_{n-2}}...xy^{a_0}$for even values of n is equal to $x^n=yx^{n-1}$yxyand for the odd values of n is equal to $x^n=yx^{n-2}yx^2y$.

Lemma 3.8. Let n ≥ 3. By considering the types of minimal subgroups of G = P SL(2, 2^n), if the subgroup H is of type $E_{2^n}{}_nZ_{2^n}$and the subgroup K is of type $D_{2(2^n+1)}$or$D_{2(2^n-1)}$ then, there exist elementsh ∈ H, k ∈ K and g ∈ G such that Hgh = Hk.

Proof. For every integer n ≥ 3, consider the minimal subgroups and $E_{2^n}{}_nZ_{2^n-1}$andK=$D_{2(2^n+1)}$. For every elements g ∈ G, h ∈ H and k ∈ K if Hgh = Hk, then $Hghk^{-1}$=HIndeed, for every elements g, h and k from G, H and K, the element ghk^{-1} doesn't belong to H, which is a contraction, because for three elements h, k and $g^0=hkh^{-1}$ from H, K and G, $g^0hk^{-1}=(hkh^{-1})hk^{-1}$= h ∈ H. So, there exist elements h ∈ H, k ∈ K and g ∈ G such that Hgh = Hk.

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