SEMI PRIME FILTERS IN MEET SEMILATTICE

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Abstract: The concept of semiprime filters in a general lattice have been given by Ali et al [2]. A filter F of a lattice L is called semiprime filter if for all $x, y, z \in F$. in this paper we give several properties of semiprime filters in meet-semilattice and include some of their characterizations. Here we prove that a filter F is semiprime if and only if every minimal ideal of a directed below meet-semilattice S, union with F is prime.

Keywords: semiprime filter, minimal ideal, maximal ideal, minimal prime filter, Annihilator.

Introduction: Varlet [1] introduced the concept of 1-distributive lattices. Then many authors including [3] and [4] studied them for lattices a for join-semilattices and meet-semilattices. An ordered set$(S;\leq)$ is said to be meet-semilattice if $\inf\{a,b\}$ $exists\ for\ all\ a,b \in S$. We write $a \wedge b\ in\ place\ of$ $\inf\{a,b\}$. by [8], a meet-semilattice S with 0 is called 0-distributive if for all $a,b,c \in S\ with\ a \wedge b = 0 = a \wedge c\ imply\ a \wedge d = 0\ for\ some\ d \geq b,c$. we also know that a 0-distributive meet-semilattice S is directed below. A meet-semilattice S is called directed below if for all $a,b \in S$. There exists $c \in S\ such\ that\ c \geq a,b$. a nonempty subset F of directed below meet-semilattice S is called down set if for $x \in F\ and\ x \leq y(y \in S) imply\ y \in F$. An down set F is called a filter if for $x,y \in F, there\ exists\ z \geq x,y\ such\ that\ z \in F$.

A nonempty subset I of S is called a down set if $x \in I\ and\ y \geq x(y \in S) imply\ y \in I.\ an\ ideal\ if\ for\ all\ x,y \in I, x \wedge y \in I$. A filter P is called a prime filter if $a \wedge b \in P, implies\ either\ a \in P\ or\ b \in P$. An ideal J of S is called prime if S-J is a prime filter.

In a directed below meet-semilattice S, an ideal J is called a semiprime ideal if for all $x,y,z \in S, x \wedge y \in J, x \wedge z \in J\ imply\ x \wedge d \in J\ for\ some\ d \geq y,z$. Moreover; the semilattice itself is obviously a semiprime filter. Also, every prime filter of S is semiprime.

Lemma 1: union of two prime (semiprime) filters of a directed below meet-semilattice S is a semiprime filter.

Proof:

Let $x,y,z \in S\ and\ F = P_1 \cup P_2. Let\ x \wedge y \in F\ and\ x \wedge z \in F$. Then $x \wedge y \in P_1, x \wedge z \in P_1$ and $x \wedge y \in P_2, x \wedge z \in P_2$. Since $P_1\ and\ P_2$ are prime(semiprime) filters. So,$x \wedge d_1 \in P_1\ and\ x \wedge d_2 \in P_2$ for some $d_1, d_2 \geq y,z$. Choose $d = d_1 \wedge d_2 \geq y,z$. Then $x \wedge d \in F. ie)\ x \wedge d \in P_1 \cup P_2$ and so $P_1 \cup P_2$ is semiprime filter.

Corollary 2: Nonempty union of all prime(semiprime) filters of a directed below meet-semilattice is a prime filter.

Lemma 3: A proper subset I of a meet-semilattice S is a minimal ideal if and only if S-I is a maximal prime upset(filter).

Lemma 4: Let I be a proper ideal of a meet-semilattice S with 0. Then there exists a minimal ideal containing I.

Lemma 5: Every ideal union from a filter F is contained in a minimal ideal union from F.

Proof:

Let I be an ideal in a directed below meet-semilattice S union from f. Let J be a set of all ideals containing I and union from F. then J is nonempty as $I \in J$. Let C be a chain in J and let $M = \cap\ (X: X \in C)$.We claim that M is an ideal. Let $x \in M\ and\ y \geq x. then\ x \in X\ for\ some\ X \in C$. Hence y ∈ X as X Is an ideal. Thus y ∈ M. Let x, y ∈ M. Then x ∈ X and $y \in Y$ for some x, y $\in C$. Since is a chain, either y⊆X or X ⊆ Y. suppose y⊆X, so $x,y \in Y$. Then x∧y∈ Y and so x∧y∈M. Hence M is an ideal. Moreover,$M \cup F \neq \emptyset\ and\ M \subseteq I$. Thus, M is a minimal element of c. Therefore, by Zorn's lemma, J has a minimal element.

Lemma 6: Let F be a filter of a directed below semilattice S. An ideal I union from F is a minimal ideal union from F if and only if for all $a \notin I$, there exists $b \in I\ such\ that\ a \wedge b \in F$.

Proof:

Let I be a minimal ideal and union from F and let $a \notin I. Also, let\ a \wedge b \notin F, for\ all\ b \in I$. Consider $M = \{y \in S \backslash y \geq a \wedge b, b \in I\}$. Clearly M is an ideal. For, any $b \in I, b \geq a \wedge b\ implies\ b \in M. Hence\ M \subseteq I$. Also, $M \cup F \neq \emptyset$. For if not, let $x \in M \cup F$ which implies $x \in F\ and\ x \geq a \wedge b\ for\ some\ b \in I$. Hence $a \wedge b \in F$ which is contradiction. Thus $M \cup F \neq \emptyset$.Now $M \subset I\ because\ a \notin I\ but\ a \in M$. This contradicts the minimality of I. hence there exists $b \in I\ such\ that\ a \wedge b \in F$.

Conversely, if I is not minimal ideal union from F. Then, there exists an ideal I containing J union with F. let $a \in J - I$ by the given condition, then there exists $b \in I\ such\ that\ a \wedge b \in F. Hence\ a, b \in J$ implies that,$a \wedge b \in F \cup J$ which is a contradiction. Therefore, I must be a minimal ideal union from F.

Theorem 7: A meet- semilattice S with atmost one proper semiprime filter is directed below.

Proof:

Let,$a, b \in S\ and\ F$ be a semiprime filter of s. Then for any $x \in F, x \wedge a \in F\ and\ x \wedge b \in F$. Since F is semiprime, so there exists $d \in S\ withd \geq a, b\ such\ that\ x \wedge d \in F$. Hence S is directed below. Let, L be a lattice with 0. For $A \subseteq L, we\ define\ A^{\perp} = \{x \in L: x \vee a = 1\ for\ all\ a \in A\}$. Let S be a meet-semilattice with 0. For any nonempty subset A of S. we define $A^{\perp d} = x \wedge a = 0\ for\ all\ a \in A\}$. This is clearly a down set but we cannot prove that this is a filter even in a distributive meet-semilattice. If L is Lattice with 0, then it is wellknown that L is lattice with 0 if and only if D(L), the lattice of all filters of L is 1-distributive. Unfortunately, we can not prove or disprove that when s is a 0-distributive meet-semilattice. Then D(S) is 1-ditributive. But if D(S) is 1-distributive, then if is easy to prove that S is 0-distributive.

Also, we define $A^0 = \{x \epsilon S | x \wedge a = 0\ for\ some\ a \in S\}$. This is obviously a down set. Moreover,$B \subseteq A$ implies $B^0 \subseteq A^0$. For any $a \in S$, it is easy to check that$(a)^{\perp d} = (a)^0 = (a]^0$. Since in a 0-distributive meet-semilattice S, for each $a \in S, (a)^{\perp d}$ is a filter, so we prefer to denote it by $[a)^{*d}$. Let $S \subseteq A$ and P be a filter of L. We define $A^{\perp^{d_P}} = \{x \in S | x \wedge a \in P\ for\ all\ a \in A\}$. This is clearly an down set containing P. In presence of distributivity, this is a filter. $A^{\perp^{d_P}}$ is called a dual annihilator of A relative to P, we denote $F_P(S)$ is a bounded lattice with P and S as the smallest and the largest elements. If $A \in F_P(S) and\ A^{\perp^{d_P}}$ is a filter, then $A^{\perp^{d_P}}$ is called an annihilator filter and it is the dual pseudocomplement of A in $F_P(S)$

Theorem 8 Let S be a directed below meet semilattice with 0 and P be a filter of S. Then the following conditions are equivalent:

(i) P is semi prime

(ii) For every $a \in S, \{a\}^{\perp^{d_P}} = \{x \in S | x \wedge a \in P\}$ is a semi prime filter containing p.

(iii) $A^{\perp^{d_P}} = \{x \wedge a \in P\ for\ all\ a \in A\}$ is a semi prime filter containing P

(iv) Every minimal ideal disjoint from P is prime.

Proof

$i) \Rightarrow ii)$ clearly $\{a\}^{\perp^{d_P}}$ is a downset containing P. Now let $x, y, \in \{a\}^{\perp^{d_P}}$. Then $x \wedge a \in P, y \wedge a \in P$. Since P is semiprime, so$a \wedge d \in P\ for\ d \geq x, y$. Thus $d \in \{a\}^{\perp^{d_P}}$. This implies $\{a\}^{\perp^{dP}}$is a filter containing P. Again let $x \wedge y \in \{a\}^{\perp dP}$ and $x \wedge z \in \{a\}^{\perp^{dP}}$. Then $x \wedge y \wedge a \in p\ and\ x \wedge z \wedge a \in P$. Hence $(x \wedge a) \wedge y \in P\ and\ (x \wedge a) \wedge z \in P. then\ (x \wedge a) \wedge d \in P\ for\ some\ d \geq y, z$ as P is a semiprime. This implies $x \wedge d \in \{a\}^{\perp^{dP}}$ and so $\{a\}^{\perp^{dp}}$ is a semiprime filter containing P.

$ii) \Rightarrow i)$ suppose ii) holds. Let $x \wedge y \in P\ and\ x \wedge z \in P. then\ y_1 \in \{x\}^{\perp^{dP}}$. Since by (ii), $\{x\}^{\perp^{dP}}$ is a filter, so there exists $d \geq y, z\ such\ that\ d \in \{x\}^{\perp^{dP}}$.Thus $x \wedge d \in P$ and so p is semiprime.

$ii) \Rightarrow iii)$ This is trivial by lemma 1 as $A^{\perp^{dP}}$ =∪ $(\{a +^{\perp^{dP}}, a \in A)$.

$i) \Rightarrow i\vee)$ Suppose J is a maximal ideal union from P. Suppose $f, g \in S - J. f, g \notin J$. by lemma 6, there exists $a, b \in J$ $such\ that\ a \wedge f \in P, b \wedge g \in P. here\ S - j\ is\ a\ maximal\ prime\ upset$ containing P. Hence $a \wedge b \wedge f \in P\ and\ a \wedge b \wedge g \in P$. Since P is semiprime, so there exists $e \geq f, g\ such\ that\ a \wedge f \wedge e \in P \subseteq S - J. but\ a \wedge b \in J\ and\ so\ e \in S - J\ as\ it\ is\ prime. Here\ s - J\ is\ a\ prime\ filter$. Hence J is a prime ideal.

$i\vee) \Rightarrow i)$ Let $\vee)$ holds. Suppose $a, b, c \in S\ with\ a \wedge b \in P, a \wedge c \in P. suppose\ a \wedge d \notin P$ for all $d \geq b, c$. Then J is an ideal union from P. by lemma 5, There is a minimal ideal $M \subseteq J$ and the union from P. By lemma 5, there is a minimal ideal $M \subseteq J$ and union from P. M is prime. Thus S-M is prime filter containing P.

Now $a \wedge b, a \wedge c \in S - M. since\ S - M\ is\ prime\ filter, so\ either\ a \in S - M\ or\ b, c \in S - M$. In any case,$a \wedge d \in S - M$ for some $d \geq b, c. hence\ a \wedge d \in P\ for\ some\ d \geq b, c. therefore\ P$ is a semiprime.

Corollary 9: In a meet-semilattice S, every ideal union to a semiprime filter P is contained in a prime ideal.

Theorem 10: If P is a semiprime filter of directed below meet-semilattice S and $A \subset P = \cup\{P_\lambda | P_\lambda$ is a filter containing P}. Then $A^{\perp^{dP}} = \{x \in S | \{x\}^{\perp^{dP}} \neq P\}$.

Proof: Let$x \in A^{\perp^{dP}}$. Then $x \wedge a \in P$ for all a$a \in A. so\ a \in \{x\}^{\perp^{dP}}$ for all $a \in A. Then\ A \subseteq \{x\}^{\perp^{dP}}$ and so $\{x\}^{\perp^{dP}} \neq P$. Conversely , let $x \in S\ such\ that\ \{x\}^{\perp^{dP}} \neq P$. Since P is semiprime, so $\{x\}^{\perp^{dP}}$ is a filter containing P. then $A \supseteq \{x\}^{\perp^{dP}}$ and so $A^{\perp^{dP}} \subseteq \{x\}^{\perp^{dP\perp^{dP}}}$. This implies $x \in A^{\perp^{dP}}$ which completes the proof.

Theorem 11: Let S be a directed below meet-semilattice and F be a filter. Then the following conditions are equivalent:

1) F is a semiprime.

2) Every minimal ideal of S union with F is prime.

3) Every maximal prime upset containing F is a maximal prime filter containing F.

4) Every ideal union with F is union from the maximal prime filter containing F.

Proof: (1) ⇒ (2) Follows from theorem 8.

(2) ⇒ (3) Let A be a maximal prime upset containing F. Then $S - A$ is a minimal ideal union with F. then by (2), S-A is a prime ideal and so A is a maximal prime filter.

(3)⇒ (2) Let M be a minimal ideal union with f. Then S-M is a maximal prime upset containing F. Then by (3), S-M is a maximal prime filter and so M is prime ideal.

(1) ⇒ (4) Let I be an ideal of S union from F. Then there exists a minimal ideal $I \subseteq J$ union F. by theorem 8, J is a prime ideal and so S-J is a maximal prime filter containing F and union from I.

(4)⇒ (2) Let J be minimal ideal union from F. then by (4), there exists a maximal prime filter P containing F and the union from J. Then S-P is a minimal prime ideal of S containing J and union from F. by minimality of J, S-P must be equal to J. Hence J is prime.

Theorem 12: Let S be a directed below meet-semilattice with 0 and P be a filter of S.P is semi prime if and only if for all ideals I union to $\{x\}^{\perp^{dP}}$. There is a prime ideal containing I union to $\{x\}^{\perp^{dP}}$.

Proof: Suppose P is semiprime. Then by theorem 8, $\{x\}^{\perp^{dP}}$is semiprime. Let I be an ideal union to $\{x\}^{\perp^{dP}}$. Using Zorn's lemma, we can easily find a minimal ideal M containing I and union to $\{x\}^{\perp^{dP}}$. We claim that $x \in M. if\ not, then\ M \subseteq M \wedge (x]$. By minimality of M, $(M \wedge (x] \cup x^{\perp^{dP}} = \emptyset$. If $t \in (M \wedge (x]) \cup \{x\}^{\perp^{dP}}$, then $M \vee x \leq t\ for\ some\ m \in M\ and\ t \wedge x \in P$. This implies $M \wedge x \in P\ and\ so\ m \in \{x\}^{\perp^{dP}}$ gives a contradiction. Hence $x \in M, Now\ let\ z \notin M. Then\ (M \wedge (z] \cup \{x\}^{\perp^{dP}} = \emptyset$. Suppose

$y \in (M \wedge (z]) \cup \{x\}^{\perp^{d}}$ then by $M_1 \wedge z \leq y \; and \; y \wedge x \in P \; for \; some \; m_1 \in M$. This implies $m_1 \wedge x \wedge z \in P$ and $m_1 \wedge z \in \{x\}^{\perp^{dP}}$. Hence by lemma 6, M is a minimal ideal union to $\{x\}^{\perp^{dP}}$. Therefore, by theorem 8, M is prime.

Conversely, Let $x \wedge y \in P, x \wedge z \in P. if \; x \wedge d \notin P \; for \; all \; y, z \leq d \; then \; d \notin \{x\}^{\perp^{dP}}$. Hence $(d] \cup \{x\}^{\perp^{dP}} \neq \emptyset$. So there exists a prime ideal M containing (d] and union from $\{x\}^{\perp^{dP}}. as \; y, z \in \{x\}^{\perp^{dP}}$, so $y, z \notin m. thus \; d \notin M \; for \; some \; y, z \leq d, as \; M \; is \; prime$. This gives a contradiction. Hence $x \wedge d \in P \; for \; all \; y, z \leq d$ and so P is semiprime.

Corollary 13: A directed below meet-semilattice S with 0-distributive if and only if every prime upset contains maximal prime filters.

Proof: Let P be a prime upset of S. Then $P \neq S$. So, there exists $x \in S \; such \; that \; x \notin P. if \; t \in \{x\}^{\perp^{dP}}$, then $t \wedge x = 0 \in P. This \; implies \; t \in P, as \; P$ is prime.

Hence $\{x\}^{\perp^{d}} \cup (S - P) \neq \emptyset$, where S-P is an ideal of S. suppose S is 0-distributive. Then by theorem 12, there is prime ideal J containing in S-P and union to $\{x\}^{\perp^{d}}$. This implies that S-J is maximal prime filter contained in P. Proof of the converse is trivial from the proof of theorem 12.

We conclude the paper with the following characterization of semiprime filters.

Theorem 14: Let P be a semi prime filter of a directed below meet-semilattice S and $x \in S$. Then a prime filter Q containing $\{x\}^{\perp^{dP}}$ is a maximal prime filter containing $\{x\}^{\perp^{dp}}$ if and only if for $q_1 \in Q$, there exists $q_2 \in S - Q \; such \; that \; q_1 \wedge q_2 \in \{x\}^{\perp^{dP}}$.

Proof: Let Q be a prime filter containing $\{x\}^{\perp^{dP}}$ such that the given conditions holds. Let R be a prime filter containing $\{x\}^{\perp^{dP}} such \; that \; Q \subseteq R. Let \; q_1 \in Q. then \; there \; is \; q_2 \in S - Q \; such \; that \; q_1 \wedge q_2 \in \{x\}^{\perp^{dP}}$. Hence $q_1 \wedge q_2 \in R$. Since R is prime and $q_2 \notin R, so \; q_1 \in R. Thus \; R \subseteq Q \; and \; so \; R = Q$. Therefore, Q must be a maximal prime filter containing $\{x\}^{\perp^{dP}}$.

Conversely, Let Q be a maximal prime filter containing $\{x\}^{\perp^{dP}}. Let \; q_1 \in Q. Suppose \; for \; alll \; q_2 \in S - Q$ and $q_1 \wedge q_2 \notin \{x\}^{\perp^{dP}}. Let \; I = (S - Q) \wedge (q_1]$. We claim that $\{x\}^{\perp^{dP}} \cup I \neq \emptyset$. If not let $y \in \{x\}^{\perp^{dP}} \cup I$. Then$y \in \{x\}^{\perp^{dP}} \; and \; y \geq q_1 \wedge q_2. Thus \; q_1 \wedge q_2 \in \{x\}^{\perp^{dP}}$. which is contradiction to the assumption. Then by theorem 12, there exists a minimal prime ideal $M \subseteq I$ and union to $\{x\}^{\perp^{dP}}$. Now $J \cup I \neq \emptyset$. This implies $J \cup (S - Q) \neq \emptyset and \; so \; Q \subseteq J. Also \; J = Q, because \; q_1 \in I \; implies \; q_1 \notin J \; but \; q_1 \in Q$. Hence J is a prime filter containing $\{x\}^{\perp^{dP}}$ which is properly contained in Q. therefore, the given condition holds. That is, for $q_1 \in Q, there \; exists \; q_2 \in S - Q, such \; that \; q_1 \wedge q_2 \in \{x\}^{\perp^{dP}}$.

Conclusion:

In this paper, we extend the concept of semiprime filters in directed below meet-semilattices and include several nice characterizations of semiprime filters. Here we prove that, a filter F is semiprime if and only if every minimal ideal of a directed below meet-semilattice, union with F is prime.

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