Stability Analysis of Linear Sylvester System on Time Scale Dynamical Systems – A New Approach

Lakshmi Vellanki, N.[1], Sriram Bhagavathula[2], Madhu J[3]

[1]*Department of Mathematics, GIS, GITAM, Visakhapatnam, AP, India, 530047*
[2]*Conns Appliances Inc*
[3]*Department of Mathematics, RGUKT, Nuzvid, AP, India, 521202*
Key words: Sylvester System, First Order Systems, Time Scale Dynamical Systems, Variation of Parameters, Stability and Asymptotic Stability, (Φ, Ψ) Bounded Functions, (Φ, Ψ) – Lebesgue Integrable functions and Fundamental Matrix Solutions.
AMS (MOS) Classifications : 34A30, 34B15, 93B05, 93B07.

---***---

Abstract : *This paper presents a criteria for the existence of (Φ, Ψ) bounded solutions of linear Sylvester system on time scale dynamical systems. The general solution for the non-homogeneous Sylvester linear system is also established. We first show that, (Φ, Ψ) bounded solution ensures the stability and asymptotic stability of the Sylvester system. It may be noted if the linear system is Φ-bounded, then the linear system T' = A(t)T is also stable and if the system T' = B*(t)T is Ψ-bounded, then the linear system is also stable*.

1. INTRODUCTION

Ψ- bounded solutions of linear system y' = A(t)y is an interesting area of current research and a great of work has been done by many authors in recent years [4-14]. Further these results have been successfully applied to fuzzy linear systems and Kronecker product systems by Kasi Viswanadh, K. N. Murty, Yan Wu, P. V. S. Anand et al., and Lakshmi vellanki N. et al., [6-15]. The importance of Ψ- bounded solution has gained momentum due to the fact that the given linear system may not be stable but the Ψ- bounded system is stable and in fact asymptotically stable. For instance, if we consider the linear system

$$y' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} y\,,$$

then a fundamental matrix of the system is given by $Y(t) = \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix}$.

As $t \to \infty, \|y(t)\| \to \infty$ and hence the given system is not stable.

Let $\psi(t) = \begin{pmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{pmatrix}$. Then

$$\psi(t)Y(t) = \begin{pmatrix} e^{-t} + e^{-2t} & 1 + 2e^{-t} \\ -2e^{-t} - 3e^{-2t} & -2 - 6e^{-t} \end{pmatrix}.$$

$\|\psi(t)Y(t)\| \le M$ and hence the system is bounded and is not in fact asymptotically stable (as $t \to \infty, \|\psi(t)Y(t)\|$ *does not tend to* 0). For this reason the importance of Ψ- bounded solutions is gaining much more attention and hence many papers came in existence during the last two years. Majority of the contributions are due to Kasi Viswanadh, K. N. Murty, Yan wu, Sailaja [5,6,7,12,13,14,19]. Recently Kasi Viswanadh, Yan wu, K. N. Murty established

(Φ, Ψ) bounded solutions on Sylvester first order non – homogeneous system [15]. Motivated by these results, we establish (Φ, Ψ) bounded solutions of linear Sylvester system on time scale dynamical system as it unifies both continuous and discrete systems. This paper is organized as follows. In section 2, we present a brief description on time scale calculus and in section 3, we present Ψ-bounded solution of linear first order system. Our main results are established in the last section. In section 2, we also establish variation of parameters formula for non-homogeneous Sylvester system on time scales which itself is new and is not available in literature.

2.1: Preliminaries:

In this section, we shall be concerned with the basic notations on time scale dynamical systems. For an excellent survey on time scale dynamical systems, we refer Lakshmi Kantham et. al., [17] and Martin Bohner and Alan Peterson [16]. By a time scale T, we mean a closed subset of R and examples of time scales are the set of natural numbers N, real numbers R, contours set etc. The set q and $A = \{t \in R|0 \leq t \leq 1\}$ are not time scales. Note that time scales need not be connected. To overcome this difficulty, we introduce the notion of Jump operators.

Definition 2.1: The operator $\sigma: T \rightarrow T$ defined by [17]

σ(t) = inf $\{s \in T: s > T\}$
ρ(t) = sup $\{s \in T: s < T\}$

are Jump operators. If σ is bounded above and ρ bounded below, then

σ(maxT) = max T and ρ(minT) = min T.

Definition 2.2: A point $t \in T$ is said to be right dense if σ(t) = t, right scattered if σ(t) > t, left dense if ρ(t) = t and left scattered if ρ(t) <t.

Definition 2.3: The graininess $\mu: T \rightarrow [0, \infty)$ is defined as $\mu(t) = \sigma(t) - t$.

We say that f is not continuous, if it is continuous in right dense points and if lim f(s) exists as $s \rightarrow t$ exists for all right dense points $t \in T$.

A function $f: T \rightarrow T$ is said to be differentiable at $t \in T$, if $\lim_{\sigma(t) \rightarrow s} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$, where $s \in T - \{\sigma(t)\}$ exists and is said to be differentiable on T for each $t \in T$. A function $f: T \rightarrow T$ with $F^{\Delta}(t) = f(t)$

For all $t \in T$, is said to be integrable, if

$\int_0^t F(t) \Delta t = F(t) - F(s)$, where f is anti delta derivative of F.

Let $f: T \rightarrow T$ and T = R and a, b ∈ T, then $F^{\Delta}(t) = f(t)$ and $\int_a^b f(t) \Delta t = \int_a^b f(t) dt$,

Further, if T = Z (discrete case) then $f^{\Delta}(t) = \Delta f(t)$

= f(t+1) – f(t) and

$$\int_a^b f(t) \Delta t = \begin{cases} \sum_{k=a}^{b-1} f(k), if\ a < b \\ 0, if\ a = b \\ \sum_{k=a}^{a-1} f(k), if\ a > b \end{cases}$$

Note that if f is Δ - differentiable, then f is continuous. Further if t is right scattered and f is continuous at t, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

For a complete survey on calculus of Δ differentiable functions, we refer to Lakshmikantham et. al., [17].

In the year 1994 Murty and Rao [18] established existence and uniqueness criteria for a two point boundary value problem on time scale dynamical system associated with

$y^{\Delta}(t) = A(t)y(t) + b(t)$

using variation of parameters formula for time scale dynamical system.

Definition 2.4: A function $F: R \to R^{n^2}$ is said to be Φ bounded on R if Φ(t)F(t) is bounded on R i.e., $\sup_{t\in R}\|\emptyset(t)F(t)\| < \infty$.

Definition 2.5: A matrix function $Y: R \to R^{n^2}$ is said to be Φ bounded on R if the matrix ΦY is bounded on R i.e., there exists M > 0 such that $\sup_{t\in R}\|\emptyset(t)Y(t)\| \le M$.

Definition 2.6: A matrix function $Y: R \to R^{n^2}$ is said to be Φ-integrable on R component wise if the matrix Φ(t)Y(t) is integrable on R i.e., $\int_0^\infty \|\emptyset(t)Y(t)\|\, dt < \infty$.

2.2: Variation of parameters formula:

In this section, we first establish variation of parameters formula for non-homogeneous Sylvester system,
$T^\Delta(t) = A(t)T(t) + T(t)B(t) + F(t)$ (2.1)

Where, A, B, T and F are all square matrices of order n and Δ differential is the

time scale derivative. The general solution of the linear system of differential equation on time scale dynamical system

$y^\Delta(t) = A(t)y(t) + C(t), y(t_0) = y_0$, (2.2)

is given by $y(t) = \emptyset(t,t_0)y_0 + \int_{t_0}^{t} \emptyset(t)\emptyset^{-1}(\sigma(s))f(s)ds$

Where, $\emptyset(t,t_0) = \emptyset(t)\emptyset^{-1}(t_0)$ is a fundamental matrix of the homogeneous system of (2.2) i.e.,

$y^\Delta(t) = A(t)y(t)$ (2.3)

Let Y(t) be a fundamental matrix solution of

$T^\Delta(t) = A(t)T(t)$

And z(t) be a fundamental matrix solution of

$T^\Delta(t) = B^*(t)T(t)$ (2.4)

Then any solution of $T^\Delta(t) = A(t)T(t) + T(t)B(t)$ is of the form T(t) = y(t)Cz*(t), where C is a constant square matrix. Such a solution cannot be a solution of the non-homogeneous system (2.1) unless $F(t) \equiv 0$. We seek a particular solution of (2.1) in the form

$\bar{T}^\Delta$ = Y(t)C(t)Z*(t).

Then $Y^\Delta(t)C(t)Z^*(t) + Y(t)C^\Delta(t)Z^*(t) + Y(t)C(t)Z^{*\Delta}(t)$

$= A(t)Y(t)C(t)Z^*(t) + Y(t)C^\Delta(t)Z^*(t)B(t) + F(t)$

Since $Y^\Delta(t) = A(t)Y(t)$ and $Z^{*\Delta}(t) = B^*(t)Z^*(t)$, we have

$Y(t)C^\Delta(t)Z^*(t) = F(t)$

Or $C^\Delta(t) = Y^{-1}(t)F(t)Z^{*^{-1}}(t)$

$$C(t) = \int_a^t Y^{-1}(\sigma(s))F(\sigma(s))Z^{*^{-1}}(\sigma(s))\,\Delta s$$

Therefore a particular solution of the non homogeneous system (2.1) is given by

$\bar{T}(t) = Y(t)\left[\int_a^t Y^{-1}(\sigma(s))F(\sigma(s))Z^{*^{-1}}(\sigma(s))\,\Delta s\right]Z^*(t)$

and hence solution T(t) of (2.1) is given by

$$T(t) = Y(t)CZ^*(t) + Y(t)\left[\int_a^t Y^{-1}(\sigma(s))F(\sigma(s))Z^{*^{-1}}(\sigma(s))\,\Delta s\right]Z^*(t) \quad (2.5)$$

Recently Kasi Viswanadh V.Kanuri, R. Suryanarayana and K. N. Murty [7] obtained existence of Ψ- bounded solutions for linear systems on time scale dynamical systems. Motivated by these results, we establish (Φ, Ψ) bounded solutions of linear Sylvester system on time scale dynamical system as it unifies both continuous and discrete systems. This paper is organized as follows. In section 2, we present a brief description on time scale calculus and in section 3, we present Ψ-bounded solution of linear first order system. Our main results are established in the last section. In section 2, we also establish variation of parameters formula for non-homogeneous Sylvester system on time scales which itself is new and is not available in literature.

$x^{\Delta}(t) = A(t)\,x(t) + f(t)$ (2.6)

where, x(t), f(t) are in T and A is a continuous matrix valued function by assuming that (2.6) has at least one Ψ- bounded solution for every Lebegue Ψ delta integrable function f on the time scale T. Note that Ψ is not a continuous matrix instead of a scalar function. In fact the problem of Ψ bounded solutions for system of ordinary differential equations has been studied recently by many authors [5-15], but a unification of these results are due to Viswanath. et al., [7]. The results on dynamical systems on measure chains has received greater impact as it includes both continuous and discrete systems in a single frame work [7]. In this paper we present a set of necessary and sufficient conditions for the existence of a (Φ, Ψ) bounded solutions for linear Sylvester system on measure chains. Recently Viswanath, Yan Wu and K. N. Murty established (Φ, Ψ) bounded solutions to a system of first order Sylvester system of differential equations. We make use of those results to establish our main results in the next section. The variation of parameters formula and its general solution given by (2.5) are in fact used as a tool to establish our results. The variation of parameters formula (2.5) is new and is interesting and tremendous applications in control engineering problems.

Theorem 2.1: Let Y(n) be a fundamental matrix solution of the linear difference system

$T(n + 1) = AT(n)$ (2.7)

and Z(n) be a fundamental matrix solution of the linear differential system

$T(n + 1) = B^* T(n)$ (2.8)

Then any solution of the difference system

$T(n + 1) = AT(n)B$ (2.9)

is of the form,

$T(n) = Y(n)CZ^*(n)$, where C is a constant scalar matrix of order n.

Proof : Consider

$T(n+1) = Y(n+1)CZ^*(n+1) = A\,Y(n)CZ^*(n)B$

$= ATB$.

Thus $Y(n)CZ^*(n)$ is a fundamental matrix of the system (2.9). Now to prove that every solution of the system (2.9) is of this form, let T be a solution and K be a matrix defined by

$K = Y^{-1}(n)\,T$

Then T = YK and

T(n + 1) = Y(n + 1)K(n + 1)

= AY(n)K(n + 1)

Since Z is a fundamental matrix of (2.8), it follows that

$Z(n + 1) = B^{*}Z(n)$

$T(n + 1) = A\,Y(n)CZ^{*}(n)B$

= ATB

Note that the result is valid only when A and B are constant square matrices of order n

We now turn our attention to the non-homogeneous Sylvester system

T (n + 1) = AT(n)B + F(n), (2.10)

then, if T(n) is a solution of (2.10) and $\bar{T}(n)$ is a particular solution of (2.10), then

T(n) - $\bar{T}(n)$ is a solution of AT(n)B and T(n) = $\bar{T}(n) + Y(n)CZ^{*}(n)$.

For T(n + 1) - $\bar{T}$(n + 1) = A[T(n) - $\bar{T}$(n)]B,

and hence T(n) = $\bar{T}$(n) + YCZ^{*}; and we seek a particular solution of (2.10) in the form

$\bar{T}(n) = Y(n)C(n)Z^{*}(n)$,

Then, $\bar{T}(n + 1) = Y(n + 1)C(n + 1)Z^{*}(n + 1)$
$= AY(n)C(n + 1)Z^{*}B$
= F(n)
Hence $C(n+1) = Y^{-1}(n)A^{-1}F(n)B^{-1}{Z^{*}}^{-1}(n)$
Thus $C(n) = \sum_{j=0}^{n-1} Y^{-1}(n-j-1)A^{-1}F(j)B^{-1}{Z^{*}}^{-1}(n-j-1)$

Hence

$$\bar{T}(n) = Y(n)\left[\sum_{j=0}^{n-1} Y^{-1}(n-j-1)A^{-1}F(j)B^{-1}{Z^{*}}^{-1}(n-j-1)\right]Z^{*}(n).$$

We shall now be concerned with the existence of a (Φ, Ψ) bounded solution of the linear Sylvester non-homogeneous system (2.1). Let R^n denote the Euclidean space of n-dimension. For $x = (x_1, x_2, x_3, x_4, \ldots x_n)^T \in R^n$,

let $\|x\| = \sup_{\|x\|\le 1}\|Ax\|$.
Let $\emptyset_i : T \to [0,\infty)$, i = 1, 2, 3, ..., n be continuous function defined by
$\Phi(t) = \mathrm{diag}[\Phi_1(t), \Phi_2(t), \Phi_3(t), \ldots, \Phi_n(t)]$,
where $\Phi_1, \Phi_2, \Phi_3, \ldots, \Phi_n$ are n-linearly independent solutions of T' = AT on T.
Definition 2.7: A function $Y: T \to R^{nxn}$ is said to be Φ bounded on T if the matrix function ΦY is bounded on T (i.e., there exists a constant M > 0 such that $\|\emptyset(t)Y(t)\| \le M$ for all t Є R or N.
Definition 2.8: A function $Y: T \to R^{nxn}$ is said to be Lebesgue Ψ integrable on T if Y is Δ measurable and ΦY is Lebesgue integrable on T.
By a solution of (2.1), we mean an absolutely continuous function YCZ^{*} satisfying (2.1) for all most all t Є T.
Let the vector space R^n be decomposed as a direct sum of the solution spaces of X_-, X_0, X_+ such that a solution T(t) of (2.1) is, Ψ bounded on T if, and only if
T(0) Є X_0 and Φ bounded on $T^{+}=[0,\infty)$ if and only if T(0) Є $X_- \otimes X_0$. Also if P_-, P_0 and P_+ denote the corresponding projections on R^n onto X_-, X_0 and X_+ respectively. It may be noted that, by a fundamental sequence in a Banach space, X, we mean a sequence in X whose span Y is dense in X. In other words, every element in X can be approximated by an element in the span. We are now in a position to present our main results in the next section.

3. MAIN RESULTS:

This section presents a criteria for the existence of (Φ, Ψ) bounded solution of the Sylvester system. If the system is (Φ, Ψ) bounded, then the system is stable. We also present a necessary and sufficient condition for the linear system to be asymptotically stable. It may be noted that by a fundamental sequence in a Banach space B, we mean a sequence in B whose span Y is dense in B. In other words every element of B can be approximated by an element in the span. That is, given x ∈ B, we can find a y ∈ Y such that $\|x-y\|$ is small.

Theorem 3.1: Let A and B be continuous (n x n) square matrices on R. Then the system (1.1) has at least one (Φ, Ψ) bounded solution R for every continuous (Φ, Ψ) bounded functions in it, if and only if,

$\int_{-\infty}^{\infty}\left\|\emptyset(t)\psi^*(t)Y(t)Z^*(t)PZ^{*^{-1}}(s)Y^{-1}(s)\psi^{*^{-1}}(s)\phi^{-1}(s)\right\|dt \le K$ (3.1)
for all $t \ge 0$, where,P = P_- on $(-\infty, t)$ and P = (P_0 + P_+) on (t, 0) and and P = P_+ on $(0,\infty)$; P = P_- on $(-\infty, 0)$ and P = (P_0 + P_-) on (0, t) and and P = P_+ on (t,∞).

Proof: First suppose that the linear system (1.1) has at least one (Φ, Ψ) bounded solution on R for every (Φ, Ψ) bounded function on R. Then it is claimed that there exists a constant K > 0 such that (3.1) holds. Let B be a Banach space of all (Φ, Ψ) bounded continuous functions $T: R \to R^{n^2}$ with norm defined by

$$\|T\|_B = \sup_{t\epsilon R}\|\phi(t)Y(t)\psi^*(t)Z^*(t)T(t)\|$$

We define
B = { $T:R^+\to R^{nxn} \times R^{nxn}$; T is Lebesgue (Φ, Ψ) integrable on R^+}
D ={ $T:R^+\to R^{nxn} \times R^{nxn}$;T is absolutely continuous on all intervals I ⊂ R^+, (Φ, Ψ) bounded on $R^+$and T(0) ∈ X_2 and T' = AT + TB } (3.2)
Clearly $C_{\Phi\Psi}$ is a real Banach space with the norm defined by
$\|T\|_{C\emptyset\psi} = \sup_{t\epsilon R^+}\|\phi(t)\psi(t)T(t)\|$

Also, it is well known that B is a real Banach space with norm

$\|T\|_B = \int_0^{\infty}\|\phi(t)\psi(t)T(t)\|dt$
The set D defined by (3.6) is a real linear space with norm
$\|T\|_D = \sup_{t\epsilon R^+}\|\phi(t)\psi^*(t)T(t)\| + \|T' - AT - TB\|_B$ is a norm on D.

Now it is claimed that $(D, \|.\|_D)$ is a Banach space. Let {T_n} be a fundamental sequence in D. Then {T_n} is a fundamental sequence in $C_{\Phi\Psi}$. Therefore, there exists a continuous and bounded function $T:R^+\to R^{nxn} \times R^{nxn}$ such that

$\lim_{n\to\infty}\phi(t)\psi^*(t)T_n(t) = \phi(t)\psi^*(t)T(t)$, uniformly on R^+.

Define $\bar{T}(t) = \psi^{*^{-1}}(t)\phi^{-1}(t)T(t)$.

Obviously $\bar{T}(t)\epsilon C\phi\psi$ and

$\|T_n(t) - \overline{T_n}(t)\| \le \left\|\psi^{*^{-1}}(t)\phi^{-1}(t)\right\|\|\emptyset(t)\psi^*(t)T_n(t) - T(t)\|$.

It then follows that $\lim_{n\to\infty}T_n(t) = \bar{T}(t)$, uniformly on every compact subset I ⊂ R^+. Thus $\bar{T}(0) \in X_2$. On the other hand {F_n(t)}, when $F_n(t) = \emptyset(t)\psi^*(t)[T_n'(t) - A(t)T(t) - B(t)T(t)]$, is a fundamental sequence in L, the Banach space of all vector functions which are Lebesgue integrable on R^+ with norm

$\|F(t)\| = \int_0^{\infty}\|\emptyset(t)\psi^*(t)F(t)\|\,dt$.

Thus there exists a function F in L such that

$\lim_{n\to\infty}\int_0^{\infty}\|F_n(t) - F(t)\| \to 0$

Replace $\bar{F}(t) = \psi^{*^{-1}}(t)\emptyset^{-1}(t)F(t)$, we get $\bar{F}(t) \in B$.

Theorem 3.2: Let A and B be (n x n) real matrices and Y(t) and Z(t) be fundamental matrix solutions of (2.3) and (2.4) respectively. Then (3.1) has at least (Φ, Ψ) bounded solution on R^+ for every Lebesgue (Φ, Ψ) integrable function F on R^+ if, and only if

$$\left\|\emptyset(t)Y(t)Z^*(t)\psi^*(t)P_1\psi^{*^{-1}}(\sigma(s))Z^{*^{-1}}(\sigma(s))Y^{-1}(\sigma(s))\emptyset^{-1}(\sigma(s))\right\| \le k \text{ for } 0 \le s \le t,$$

(3.3)

and

$$\left\|\emptyset(t)Y(t)Z^*(t)\psi^*(t)(I-P_1)\psi^{*^{-1}}(\sigma(s))Z^{*^{-1}}(\sigma(s))Y^{-1}(\sigma(s))\emptyset^{-1}(\sigma(s))\right\| \le k\text{for } 0 \le s \le t$$

(3.4)

Proof : We use the following notation
$\widehat{\emptyset}(t) = \emptyset(t)Y(t)$ and $\hat{\psi}(t) = Z^*(t)\psi^*(t)$ then, (3.3) and (3.4) reduces to
$\left\|\widehat{\emptyset}(t)\hat{\psi}(t)Z^*(t)P_1\hat{\psi}^{-1}(\sigma(s))\widehat{\emptyset}^{-1}(\sigma(s))\right\| \le k$ for $0 \le s \le t$ (3.5)
and $\left\|\widehat{\emptyset}(t)\hat{\psi}(t)(1-P_1)\widehat{\psi}^{-1}(\sigma(s))\widehat{\emptyset}^{-1}(\sigma(s))\right\| \le k$ for $0 \le t \le s$. Suppose the conditions (3.3) and (3.4) hold, then it is claimed that (3.1) has at least one (Φ, Ψ) bounded solution on R^+. This eventually proves that system (3.1) is stable. Now we define the sets $C_{\Phi\Psi}$ ={T:$R^+ \to R^{nxn}$ x R^{nxn}; T is (Φ, Ψ) bounded and continuous on R^+} For any fixed t ≥ 0,
we have $\bar{T}(t) - \bar{T}(0) = \lim_{n\to\infty}\|T_n(t) - T_n(0)\| = \lim_{n\to\infty}\int_0^t T_n'(s)ds$

$$=\lim_{n\to\infty}\int_0^\infty \left[\psi^{*^{-1}}(t)\emptyset^{-1}(t)[F_n(s) - F(s) + A(t)T_n(t) + T_n(t)B(t)]\right]\text{ds}$$

$$= \int_0^\infty [F(s) + A(s)T(s) + T(s)B(s)]\text{ds}$$

It follows that $T_n'(t) - A(t)T(t) - T(t)B(t) = \bar{F}(t) \in B$ and $\bar{T}(t)$ is absolutely continuous in all intervals I ⊂ R^+. Thus $\bar{T}(t) \in D$. From $\lim_{n\to\infty}\phi(t)\psi^*(t)T_n(t) = \phi(t)\psi^*(t)T(t)$ uniformly on R^+ and

$$\lim_{n\to\infty}\int_0^\infty\left\|\phi(t)\psi^*(t)[T_n'(t) - A(t)T(t) - T(t)B(t)] - \left[\widehat{T}'(t) - A(t)\widehat{T}(t) - \widehat{T}(t)B(t)\right]\right\|dt=0$$

It follows that $\lim_{n\to\infty}\|T_n(t) - \bar{T}(t)\|_D = 0$. Thus $K(D, \|.\|_D)$ is a Banach space and hence

$$\sup_{t\in R^+}\|\phi(t)\psi^*(t)T(t)\| \le K\int_0^\infty\|\phi(t)\psi^*(t)F(t)\|\,dt.$$

For s > 0, ? > 0, ξ Є R^n, we consider the function F: $R^+ \to R^{nxn}$

$$F(t) = \begin{cases}\psi^{*^{-1}}(t)\phi^{-1}(t)\xi, s \le t \le s+\delta \\ 0, elsewhere\end{cases}$$

Then F Є B and $\|F\|_B = \delta\|\xi\|$. The corresponding solution T Є D is,

$$T(t) = \int_s^{s+\delta} G(t,s)\,ds,$$

$$\text{Where } G(t,\sigma(s)) = \begin{cases} Y(t)Z^*(t)P_1Z^{*^{-1}}(\sigma(s))Y^{-1}(\sigma(s)), 0 \le k \le t \\ -Y(t)Z^*(t)P_2Z^{*^{-1}}(\sigma(s))Y^{-1}(\sigma(s)), 0 \le t \le k\end{cases}$$

Clearly G is continuous and possesses continuous first order partial derivatives at all points except at t = $\sigma(s)$, where G has a jump discontinuity of finite magnitude. Therefore

$$\|\phi(t)\psi^*(t)\| = \int_s^{s+\delta} \left\|\phi(t)\psi^*(t)G\big(t,\sigma(s)\big){\psi^*}^{-1}\big(\sigma(s)\big)\phi^{-1}\big(\sigma(s)\big)\right\|ds$$

$$=k\|\xi\| .$$

Hence $\left\|\phi(t)\psi^*(t)G\big(t,\sigma(s)\big){\psi^*}^{-1}\big(\sigma(s)\big)\phi^{-1}\big(\sigma(s)\big)\right\| \leq K$, which is equivalent to (3.5). Now, to prove the other part, we consider the function

$$T(t) = \int_0^t Y(t)Z^*(t)P_1{Z^*}^{-1}\big(\sigma(s)\big)Y^{-1}\big(\sigma(s)\big)F\big(\sigma(s)\big)ds$$

$$- \int_t^\infty Y(t)Z^*(t)P_2{Z^*}^{-1}\big(\sigma(s)\big)Y^{-1}\big(\sigma(s)\big)F\big(\sigma(s)\big)ds \text{ , t > 0,}$$

where F is Lebesgue (Φ, Ψ) integrable function on R^+ It can easily be proved that T is a (Φ, Ψ) bounded solution on R^+. The proof of the theorem is complete.

Theorem 3.3: The fundamental matrix solutions Y(t) and Z(t) satisfy the conditions

i) $\lim_{t\to\infty}\|\phi(t)Y(t)Z^*(t)\psi^*(t)P_1\| = 0$
ii) $\left\|\phi(t)Y(t)Z^*(t)\psi^*(t)P_1\psi^*\big(\sigma(s)\big){Z^*}^{-1}\big(\sigma(s)\big)Y^{-1}\big(\sigma(s)\big)\phi^{-1}\big(\sigma(s)\big)\right\| \leq K$ for $0 \leq s \leq t$
iii) $\left\|\phi(t)Y(t)Z^*(t)\psi^*(t)P_2\psi^*\big(\sigma(s)\big){Z^*}^{-1}\big(\sigma(s)\big)Y^{-1}\big(\sigma(s)\big)\phi^{-1}\big(\sigma(s)\big)\right\| \leq K$ for $0 \leq t \leq s$

Proof : Proof follows from the above theorem by taking $P_0 = 0$.

References:

[1] O.Akinyele, on partial stability and boundedness of degree K, Att. Acad. Naz. Linear Rend cl. Sci.Fis.Mat. Natur; Vol 65:259-264(1978)

[2] W. A. Coppel, stability and asymptotic stability behavior of solutions of differential equations, Health Boston, 1965.

[3] T. G. Hallam, On asymptotic equivalence of the bounded solutions of two systems of differential equations, Miceo. Masu Jour. Vol. 16, p:353-363(1969)

[4] A. Diamandescu, Existence of Ψ- bounded solutions for a system of differential equations, Electronic Jour. of differential equation Vol. 63, No:1, p: 1-6(2004).

[5] Kasi Viswanadh V. Kanuri, K. N. Murty, "Three-Point boundary value problems associated with first order matrix difference system-existence and uniqueness via shortest and closest Lattice vector methods", Journal of Nonlinear Sciences and Applications, Volume 12, Issue 11, (2019) 720--727

[6] Kasi Viswanadh V Kanuri, Existence Of Ψ-Bounded Solutions For Fuzzy Dynamical Systems On Time Scales, International Journal of Scientific & Engineering Research, 2020, Vol. 11, No. 5, 613--624.

[7] Kasi Viswanadh V. Kanuri, R. Suryanarayana, K. N. Murty, Existence of Ψ-bounded solutions for linear differential systems on time scales, Journal of Mathematics and Computer Science, 20 (2020), no. 1, 1--13.

[8] Murty, K. N., Andreou, S., Viswanadh, K. V. K., Qualitative properties of general first order matrix difference systems. Nonlinear Studies, 16(2009), no. 4, 359--370.

[9] K. N. Murty, V. V. S. S. S. Balaram, K.V. K. Viswanadh, "Solution of Kronecker Product Initial Value Problems Associated with First Order Difference System via Tensor-based Hardness of the Shortest Vector Problem", Electronic Modeling, vol. 6, p: 19--33 (2008).

[10] Kasi Viswanadh V Kanuri, Yan Wu, Sriram Bhagavathula, Existence of (Φ, Ψ)- bounded solutions of Linear Sylvester system of differential equations. International Journal of Scientific & Engineering Research Volume 11, Issue 10, PP: 995 – 1005.

[11] K.N. Murty, Yan Wu, Viswanadh V. Kanuri, "Metrics that suit for dichotomy, well conditioning of object oriented design", Nonlinear Studies, Vol 18, No: 4, P: 621--637 (2013)

[12] Kasi Viswanadh V Kanuri, Y. Wu, K.N. Murty, "Existence of $(\Phi \otimes \Psi)$ bounded solution of linear first order Kronecker product of system of differential equations", International Journal of Science and Engineering Research, 2020, Vol 11, No. 6, P: 721--730.

[13] Sailaja,P. and Viswanadh,K. V. K. and Murty,K. N.,"A new approach to the construction of transition matrix with application to control systems", 2020, AIP Conference Proceedings, volume 2269, number 1,pp 030044.

[14] Lakshmi vellanki N., N. Vijaya, Madhu J, Existence of $(\emptyset^{\alpha} \otimes \psi^{\alpha})$ bounded solutions of Kronecker product first order system of Differential equations, Int. Jour. Of Engineering and Comuter Science, Vol. 9, No:10, p:25240- 25245(Oct2020)

[15] Kasi Viswanadh V. Kanuri, Sriram Bhagavathla, K. N. Murty, A New Approach to the Stability Analysis of Linear Sylvester System of First Order Differential Equations, International Journal of Engineering and Computer Science Vol. 9 No. 11 (2020) | Page No.: 25252-25259.

[16] Martin Bohner, Allan Peterson, "Dynamical equations on time scales", Birkhauser, Boston (2003).

[17] V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, Dynamic systems on measure chains, Kluwer Academic Publishers Group, Dordrecht, (1996).

[18] K.N. Murty, Rao,Y.S, Two point-boundary value problems on inhomogeneous Timescale dynamical systems, Jour.Math.Analysis and Applications, Vol: 184, P: 22-34 (1994) .

[19] Y. Wu, D.L. Nethi, K.N.Murty, Initial Value Problems Associated With First Order Fuzzy Difference System -Existence and Uniqueness, International Journal of Recent Scientific Research. 11 (2020), no. 3, pp 37846--37848.