A Literature Review of Convergence Rate Analysis of Gaussian Belief Propagation

Swathi K[1], Binesh K[2]

[1]M.Tech student, Dept. of Electronics &Communication Engineering, College of Engineering Thalassery
[2]Assistant Professor, Dept. of Electronics &Communication Engineering, College of Engineering Thalassery

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Abstract - *Gaussian belief propagation algorithm (GaBP) is one of the most important distributed algorithms in signal processing and statistical learning involving Markov networks. It is known that the algorithm correctly computes marginal density functions from a high dimensional joint density function over a Markov network in a finite number of iterations when the underlying Gaussian graph is acyclic. Analysis of convergence rate is an important factor. Different methods for analysing convergence rate is presented in this paper. GaBP algorithm is shown to converge faster than classical iterative methods like Jacobi method, successive over relaxation. It is more recently known that walk summability approach extends for better convergence result.*

Key Words: Belief propagation, distributed algorithm, Gaussian belief propagation, Markov network, convergence rate.

1. Introduction

Belief propagation (BP) algorithm is a well-celebrated distributed algorithm for Markov networks that has been widely utilized in many disciplines, ranging from statistical learning and artificial intelligence to distributed estimation distributed optimization, networked control and digital communications. It is designed compute the marginal probability densities of random variables from the joint probability density function over a large Markov network with sparse connections among individual random variables. The Gaussian BP algorithm (GaBP), a special version of the BP algorithm for Markov networks with Gaussian distributions (also known as Gaussian graphical model), has received special attention for the study of its convergence properties. This paper discusses about the analysis of convergence of rate of GaBP under different application. And also points out the best method for analysis.

2. Literature Review

Bin Xia, Wenhao Yuan, Nan Xie, and Caihong Li develop a novel statistical manifold algorithm for position estimation in wireless sensor networks. It ables to find distance information among unknown and anchor nodes, in following steps: First, a ranging model about distance information is established. Then a solution problem to this established model, it is then transformed into a parameter estimation of curved exponential family.Next a solution to this estimation problem is obtained by natural gradient method. To ensure convergence of the proposed algorithm, a particle swarm optimization method is utilized to obtain initial values of the unknown nodes. PSO is a computational method that optimizes a problem iteratively.

In [2], convergence rate is analysing using Gaussian Belief Propagation Solver(GaBP).This paper describes an undirected graphical models corresponding to the linear systems of equations. Applying Belief Propagation in graphical model and updating the BP equation using performed. In ordinary BP, convergence does not guarantee exactness of the inferred probabilities, unless the graph has no cycles. Its underlying Gaussian nature yields a direct connection between convergence and exact inference. The following two theorems establish sufficient conditions under which GaBP is guaranteed to converge to the exact marginal means.

i. If the matrix A is strictly diagonally dominant (i.e, $|A_{ii}| > \sum_{j\neq i}|Aij|,\forall i$), then GaBP converges and the marginal means converge to the true means. This sufficient condition was recently relaxed to include a wider group of matrices.

ii. If the spectral radius (i.e. , the maximum of the absolute values of the eigen values) ρ of the matrix $|I_n - A|$ satisfies $\rho(|I_n - A|) < 1$, then GaBP converges.

In 2015, Qinliang Su and Yik-Chung Wu proposed different convergence conditions for Gaussian Belief Propagation[3].The complexity of directly computing the marginal PDF will be very high. So by passing messages from neighbouring nodes in factor graph, BP provides an efficient way to compute the approximate marginal PDFs upon convergence. This paper deals with describing the message passing process of GaBP on pairwise factor graph as a set of updating function. The convergence conditions of beliefs for synchronous Gaussian BP, damped Gaussian BP and asynchronous Gaussian BP are derived:

i.In synchronous Gaussian BP, belief parameters($\sigma_i{}^2(t)$,μ_i(t)) converge to the same point for all choices of $v^a(0) \in \mathcal{A}$ and $\beta^a(0) \in \mathbb{R}^{|\varepsilon|}$ if and only if $S_1 \neq \emptyset, \rho(G^*) < 1$ andp_{ii}+$\sum_{k\in\mathcal{N}(i)} w_{ki}^* \neq 0$ where v^a , β^a are parameters.

1) $S_1 \triangleq \{w|w \leq g(w) and\ w\epsilon\mathcal{W}\}$,$\mathcal{A} \triangleq \{w \geq 0\} \cup \{w \geq w_0|w_0\epsilon\ int\ (S_1)\} \cup \{w \geq w_0|w_0 \in S_1\ and\ w_0$=$\lim_{t\to\infty} g^{(t)}(0)\}$ .ii.DampedGaBP, belief parameters($\sigma_i{}^2(t)$,μ_i(t)) converge to the same point for all choices of $v^a(0) \in \mathcal{A}$ and $\beta^a(0) \in \mathbb{R}^{|\varepsilon|}$ under a nonzero damping factor d if and only if the three conditions hold:1) $S_1 \neq \emptyset$,
2) $\max_{\lambda(G^*)} \Re\left(\lambda(G^*)\right) < 1$or $\min_{\lambda(G^*)} \Re(\lambda(G^*)\ > 1$;
3) p_{ii}+$\sum_{k\in\mathcal{N}(i)} w_{ki}^* \neq 0$
iii.AsynchronousGaBP,
if $S_1 \neq \emptyset, \rho(|G^*|) < 1$and

p_{ii}+$\sum_{k\in\mathcal{N}(i)} w_{ki}^* \neq 0$,belief parameters($\sigma_i{}^2(t)$,$\mu_i$(t)) converge to the same point for all choicesof$v^a(0) \in \mathcal{A}$and$\beta^a(0) \in \mathbb{R}^{|\varepsilon|}$ Convergence condition for asynchronous GaBP is more stringent than that of synchronous GaBP.

In[4],Patrick Rebeschini and Sekhar Tatikonda investigates the behaviour of the min-sum message passing scheme to solve Laplacian matrices of graphs and to compute systems of linear equations in the electric flow.

Yair Weiss and William T.Freeman [5], analyzes belief propagation in network with arbitrary topologies when the nodes in the graph describes jointly Gaussian random variables .Then giving an analytical formula relating the true posterior probabilities. Sufficient condition for convergence is given and shows that when belief propagation converges it gives correct posterior means for all graph topologies. The performance of belief propagation in general networks with multiple loops is carried out.The sum-product and max-product belief propagation algorithms are appealing, fast and easily parallelizable algorithms. The results give a theoretical justification for applying belief propagation in certain networks with multiple loops. This may enable fast, approximate probabilistic inference in a range of new applications.

In [6],the paper discusses about the theoretical framework for analyzing graph laplacians and operator. Analysis of graph laplacians including KNN graph. The framework reduces the problem of graph laplacian analysis to the calculation of a mean and varience for any graph construction method with positive weights and shrinking neighbourhood. It extends existing strong operator convergence results to non smooth kernels. Graphical models provide a powerful formalism for statistical signal processing. Due to their sophisticated modeling capabilities, they have found applications in a variety of fields such as computer vision, image processing, and distributed sensor networks.

In[7], a general class of algorithms for estimation in Gaussian graphical models with arbitrary structure is presented. These algorithms involve a sequence of inference problems on tractable sub graphs over subsets of variables. Analysis of algorithms based on the recently developed walk-sum interpretation of Gaussian inference. This leads to efficient methods for optimizing the next iteration step to achieve maximum reduction in error. Walk Summability approach is used for analyzing convergence rate. If every edge is updated infinitely often, then computed converges to the correct means in walk-summable models for any initial guess. Also shows that walk- summability is a sufficient condition for all algorithms to converge for a very large and flexible set of sequences of tractable subgraphs or subsets of variables on which to perform successive updates. For any non-walk-summable model, there exists at least one sequence of iterative steps that is ill-posed.

The paper [8] presents a new framework based on walks in a graph for analysis and inference in Gaussian graphical models. The key idea is to decompose correlations between variables as a sum over all walks between those variables in the graph. The weight of each walk is given by a product of edgewise partial correlations and provided a walk-sum interpretation of Gaussian belief propagation in trees and of the approximate method of loopy belief propagation in graphs with cycles. This perspective leads to a better understanding of Gaussian belief propagation and of its convergence in loopy graphs.

Let ϱ(A) denote the spectral radius of a symmetric matrix A, defined to be the maximum of the absolute values of the eigen values of A. The geometric series (I +A+A2+. . .) converges if and only if ϱ (A) < 1. If it converges, it converges to $(\mathrm{I}-A)^{-1}$. If ϱ (R) < 1, then we have a geometric series for the covariance matrix: .$\sum_{l=0}^{\infty} R^l$ = $(\mathrm{I}-A)^{-1}$=J^{-1}=P.Let $\bar{R}$ = (|rij |) denote the matrix of element-wise absolute values.The model is walk-summable if ϱ ($\bar{R}$) < 1. Walk-summability implies ϱ (R) < 1 and J≻ 0.

3. CONCLUSIONS

This article mainly discusses about different methods for analysing convergence rate of GaBP under application levels. Different methods provide different accuracy levels for convergence. As a review from those, walk summability approach provides better result. Future extension can be done using laplacian solvers and it can be undertaken for bayesian network instead of markov models.

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