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Analytic Evaluation of the Head Injury Criterion (HIC) within the Framework of Constrained Optimization Theory

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Abstract - *The Head Injury Criterion (HIC) is a weighted impulse function, based upon the measured or calculated resultant acceleration, a(t), at the head center of mass, which predicts the probability of closed head injury when coupled with an appropriate statistical model. Previous published work has elucidated a number of important findings in regards to determining the temporal segment, over the operative domain of the resultant acceleration, which maximizes the HIC when a(t) is an analytic closed-form function and when there is no constraint on the length of the temporal segment. Current formulations of the HIC, however, are predicated upon using a temporal windowing function that delimits the maximum duration over which the function is calculated. Presented in the subject work, within the framework of constrained optimization theory, are the solutions for determining the initiating and terminus values of the windowing function, for internal domain points, for the cases in which a(t) is defined by a single function over its domain and when any internal domain point is based upon two functions with differing analytic closed-form representations. The general solutions are crystalized by considering example application cases.*

Key Words: Biomechanics, Closed head injury, Brain injury, Head Injury Criterion, Constrained optimization

1. INTRODUCTION

The Head Injury Criterion (HIC) has been the primary metric employed by United States (US) National Highway Traffic Safety Administration (NHTSA), in regards to predicting the probability of closed head injury, for approximately half a century. The historical development of the HIC traces to the severity index (SI), which in turn, represents a log-linear fit to the Wayne State Tolerance Curve (WSTC). The shortcomings of the WSTC, which was developed for frontal head impacts [1-6] have been reviewed elsewhere [7-8]. The underlying issues with the SI [9] were detailed in the initial development of the HIC [10]. Despite the historical developmental shortcomings of the HIC, the current instantiation of the same has been shown to correlate well with the development, or lack thereof, of concussive head injury [11]. Mathematically, this current instantiation is expressed as the following:

$$\mathrm{HIC} = \left[(t_2 - t_1) \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} a(t)\,dt \right)^{2.5} \right]_{max} \qquad (1)$$

In equation (1), a(t) is the resultant head center of mass translational acceleration (i.e. the magnitude of the triaxial translational acceleration at the center of mass of the head). The temporal values of $t = t_1$ and $t = t_2$ $\{t_1, t_2: t_1 < t_2\}$ are within the temporal domain of a(t). The function is maximized when t_1 is equal to some value, τ_1, and t_2 is equal to some value, τ_2, within the domain of a(t) and with $\{\tau_1, \tau_2: \tau_1 < \tau_2\}$. The function a(t) is expressed in terms of Gs whereas the temporal variables are expressed in terms of seconds. The current instantiation of the HIC also involves a limitation on the maximum duration over which equation (1) is calculated. In other words, denoting d as $d = t_2 - t_1$, the value of d is such that $d \le d_{max}$. Notations such as HIC_{15} or HIC_{36} explicitly denote the maximum length of the temporal window, in terms of milliseconds, in the subscript. These formulations are referred to as clipped HIC formulations.

Chou an Nyquist, shortly after the introduction of the HIC to the biomechanical engineering corpus, presented a set of conclusions based upon the analytic evaluation of the unclipped function [12]. These conclusions included the finding that the critical values of $\{t_1, t_2\}$, denoted above as $\{\tau_1, \tau_2\}$, for the nontrivial case of $\tau_1 \ne \tau_2$, occurred where $a(\tau_1) = a(\tau_2)$ and that the average acceleration of a(t) between the two critical points was equal to five thirds of the acceleration at either critical point. The authors also presented closed-form analytic solutions for the cases in which a(t) was modeled using a half sine, triangular, trapezoidal or square pulse shape. The unclipped nature of the formulations developed by those authors can readily be shown by example. For the case of a(t) modeled using a half sine pulse shape, initiating at t = 0 and terminating at t = T, the authors determined that the critical values were $\tau_1 = 0.1651T$ and $\tau_2 = 0.8349T$. The critical duration, being the difference of these two values, is 0.6698T. The dependence on T directly indicates that the evaluation was unclipped.

To the knowledge of the subject author, the analytic evaluation of the clipped HIC function has not been addressed within the scientific literature. The objective of the author, by means of the subject work, is the development of the analytic relationships for the case of the clipped HIC function and the presentation of the results obtained by applying these relationships to certain closed form pulse shapes.

2. THEORY

2.1 Constrained optimization

The formal requirements for optimizing a function f(**x**), subject to the equality constraints $h_i(\mathbf{x}) = 0$ {i: i = 1, 2, ..., p} and inequality constraints $g_j(\mathbf{x})$ {j: j = 1, 2, ..., q} are ensconced in the first order necessary and second order sufficient Karush-Kuhn-Tucker (KKT) conditions [13]. The constraint relationships are appended to f(**x**) by means of Lagrange multipliers. The resulting Lagrangian function is:

$$L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda},\mathbf{s}) = f(\mathbf{x}) + \sum_{i=1}^{p}\lambda_i h_i(\mathbf{x}) + \sum_{j=1}^{q}\mu_j\left(g_j(\mathbf{x}) + s_j^2\right) \quad (2)$$

Where **s** is a vector of slack variables such that:

$$g_j(\mathbf{x}) + s_j^2 = 0 \quad \forall j \quad (3)$$

For the case in which the optimization is a maximization problem, the inequality constraints $g_j(\mathbf{x})$ are in the form $g_j(\mathbf{x}) \geq 0 \ \forall j$. For the maximization problem, when the j^{th} inequality constraint is active, $s_j^2 = 0$ and $\mu_j \geq 0$. When the j^{th} inequality constraint is inactive, $\mu_j = 0$ and $s_j^2 \leq 0$. The solution vector $\mathbf{x} = \xi$ that maximizes f(**x**) subject to h(**x**) and g(**x**), occurs when the derivative of the Lagrangian with respect to all of its independent variables is zero-valued. The first order necessary KKT optimality conditions consist of the four, coupled, vector equations that express this first derivative test.

$$\frac{\partial L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda},\mathbf{s})}{\partial \mathbf{x}} = \nabla f(\mathbf{x}) + \sum_{i=1}^{p}\lambda_i \nabla h_i(\mathbf{x}) + \sum_{j=1}^{q}\mu_j \nabla g_j(\mathbf{x}) = 0 \quad (4)$$

$$\frac{\partial L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda},\mathbf{s})}{\partial \boldsymbol{\mu}} = g_j(\mathbf{x}) + s_j^2 = 0 \quad \forall j \quad (5)$$

$$\frac{\partial L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda},\mathbf{s})}{\partial \boldsymbol{\lambda}} = h_i(\mathbf{x}) = 0 \quad \forall i \quad (6)$$

$$\frac{\partial L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda},\mathbf{s})}{\partial \mathbf{s}} = 2\mu_j s_j = 0 \quad \forall j \quad (7)$$

Equation (7) contains the switching conditions (i.e. complementary slackness). The second order sufficient KKT conditions, corresponding to the second derivative test, can be expressed as the following:

$$\mathbf{z}^T \begin{pmatrix} \nabla^2 f(\xi) + \sum_{i=1}^{p}\lambda_i \nabla^2 h_i(\xi) + \\ \sum_{j \in active}\mu_j \nabla^2 g_j(\xi) \end{pmatrix} \mathbf{z} > 0 \quad \forall \mathbf{z} \in \mathbb{R}^n, \mathbf{z} \neq \mathbf{0} \quad (8)$$

Where the vector **z** denotes any non-zero real valued vector and ∇^2 denotes the Hessian of its operand. It is important to note that equation (8) is evaluated at the KKT points (i.e. at $\mathbf{x} = \xi$) and only for those inequality constraints that are active. This equation is valid if the parenthetical is a positive definite matrix. The feasible changes, **z**, must satisfy the following gradient conditions for the constraints.

$$\nabla h_i(\xi)^T \mathbf{z} = 0 \quad \nabla g_j(\xi)^T \mathbf{z} = 0 \quad (9)$$

2.2 Clipped HIC as a constrained problem

Equation (1) can be rewritten by first separating the two multiplicative terms that are exponentiated to the 2.5 power and reducing the resultant two exponentiated terms involving $t_2 - t_1$. Furthermore, the term t_1 is replaced with t_a and the term t_2 is replaced with $t_a + d$, where d is the temporal length of the windowing function. The resultant is the following:

$$f(\mathbf{x}) = f(t_a,d) = HIC(t_a,d) = d^{-1.5}\left(\int_{t_a}^{t_a+d} a(t)dt\right)^{2.5} \quad (10)$$

For the case in which the resultant acceleration, a(t), is either an analytic closed-form function, or modeled as such, it is useful to define N + 1 {N: N$\in \mathbb{Z}$, N ≥ 1} transition points. These points, collectively, represent the points at which the known resultant acceleration undergoes a salient change. In the simplest case, that being one in which a(t) is monotonic function, N = 1. The first transition point occurs at $t = t_0$ and with corresponding resultant acceleration $a(t_0) = a_0$, which represents the start of the function. The second transition point occurs at $t = t_1$ and with corresponding resultant acceleration $a(t_1) = a_1$, which represents the end of the function. When a(t) is either a composite function of two temporally segmented functions or when the form of the function is explicitly based on a single value internal to the domain, N = 2. The internal point for such cases is denoted as $\{t_1, a(t_1)\}$ and the terminus point is denoted as $\{t_2, a(t_2)\}$. It can thus readily be seen that for N ≥ 1, the resultant acceleration function contains N - 1 internal transition points. There is no upper limit on the number of transition points, however, as N becomes arbitrarily large, the utility of using closed form analytic solutions decreases when compared with simple numerical evaluation.

Equation (10) contains a number of implicit constraints, which do not require reiteration in the form of the inequality constraints found within g(**x**). The first implicit constraint arises from the fact that a(t) is the resultant acceleration. Therefore, $a(t) \geq 0 \ \forall t$ {t: $t_0 \leq t \leq t_N$}. The second implicit constraint arises from the ordering of the limits of integration. Because $t_a + d > t_a$, $d > 0$ and $d^{-1.5} > 0$. When coupled with the first implicit constraint, the exponentiated integral term in equation (10) is either positive or zero valued ∀t. As a consequence of these implicit constraints, $HIC(t_a, d) \geq 0$ over the temporal domain.

In regards to the explicit constraints, there are no equality constraints and therefore $h(\mathbf{x})$ is zero-valued and can be dropped from the formulation. Three inequality constraints are considered herein. The first is the relationship between the temporal length of the windowing function that clips the HIC function, with respect to a maximal specified length (d_{max}). The inequality of $d_{max} - d \geq 0$ leads to the following:

$$g_1(t_a, d) + s_1^2 = (d_{max} - d) + s_1^2 = 0 \tag{11}$$

When this constraint is active, $s_1^2 = 0$, $\mu_1 \geq 0$ and $d = d_{max}$. When this constraint is inactive, $s_1^2 \leq 0$, $\mu_1 = 0$ and the value of d is unconstrained. The second constraint relates the first critical point to the first transition point. The inequality of $t_a - t_0 \geq 0$ leads to the following:

$$g_2(t_a, d) + s_2^2 = (t_a - t_0) + s_2^2 = 0 \tag{12}$$

When this constraint is active, $s_2^2 = 0$, $\mu_2 \geq 0$ and $t_a = t_0$. When this constraint is inactive, $s_2^2 \leq 0$, $\mu_2 = 0$. Even with t_a being unconstrained with respect to t_0 when this constraint is not active, solutions that involve $t_a < t_0$ can be excluded on an implicit basis. The third constraint relates the second critical point to the last transition point. The inequality of $t_N - t_a - d \geq 0$ leads to the following:

$$g_3(t_a, d) + s_3^2 = (t_N - t_a - d) + s_3^2 = 0 \tag{13}$$

When this constraint is active, $s_3^2 = 0$, $\mu_3 \geq 0$ and $t_a + d = t_N$. When this constraint is inactive, $s_3^2 \leq 0$, $\mu_3 = 0$. Even with $t_a + d$ being unconstrained with respect to t_N when this constraint is not active, solutions that involve $t_N < t_a + d$ can be excluded on an implicit basis.

There are a number of salient points in regards to the subject formulation of the inequality constraints that comprise $g(\mathbf{x})$. Generally, for a maximization or minimization problem, the solution approach requires checking the boundary points (i.e. $t = t_0$ and $t = t_N$) in addition to checking the internal domain (i.e. $t_0 < t < t_N$) for the value(s) of the domain that optimize the function. When N = 1, checking the first boundary point is achieved by setting the second constraint as active and checking the second boundary point is achieved by setting the third constraint as active. The case when both the second and third constraints are active is only valid, for N = 1, when $t_1 - t_0 = \Delta t_{10} = d$. The determination of the necessity of performing such a check can be determined a priori since a(t) is considered as known. Secondly, when it is known that $\Delta t_{10} < d_{max}$, the first constraint can be rewritten by replacing d_{max} with Δt_{10}. Thirdly, when the first and second constraints are active for N = 1, $t_a = t_0$ and $d = d_{max}$. This is a check for the first boundary point and with the clipping constraint set to active. Similarly, when the first and third constraints are active for N = 1, $t_a = t_N - d_{max}$ and $d = d_{max}$. This is a check for the second boundary point and with the clipping constraint set to active. Finally, the number of constraints, generally, when N > 1, must be increased in number. As an example, consider the case of N = 2. The resultant acceleration, a(t), is comprised of a function $a_1(t)$ that is active over the domain $\{t: t_0 \leq t \leq t_1\}$ and a second function $a_2(t)$ that is active over the domain $\{t: t_1 \leq t \leq t_2\}$. Each segment, again, generally, requires checking both boundary points and the internal domain. The check of $t = t_1$ as a second critical point solely involves $a_1(t)$ whereas checking the same point as a first critical point solely involves $a_2(t)$. These checks involve the inclusion of additional constraints in the form of the second and third constraint. Additionally, one must check for the case where $t_a < t_1 < t_a + d$. Therefore, for N = 2, there are seven checks involved. Generalizing this for N > 1, there are 4N – 1 checks involved. Furthermore, on a simple combinatorial basis, there exist $2^3 = 8$ combinations of the three inequality constraints, per segment. The total number of salient combinations can readily be reduced since the form of a(t) is known.

2.3 Partial derivatives of the HIC function

The implementation of the KKT conditions requires being able to differentiate the HIC function. It is instructive to first consider the case where a(t) is a single function. To simplify the form of the solutions, the shorthand notation of $a(t_a) = a_a$ and $a(t_a + d) = a_d$ are employed. Also, the difference $a_d - a_a$ is denoted as Δa_{da}. Finally, the integration of a(t) over $t = t_a$ and $t = t_a + d$ is denoted as:

$$\int_{t_a}^{t_a+d} a(t)dt = f_1(t) \tag{14}$$

Taking the partial derivative of the HIC function with respect to either t_a or d is complicated by the fact that t_a appears in both the lower and upper limits of integration and that d appears in the upper limit of integration. Differentiation under the integration is employed by using Leibniz's rule. The context specific keys to implementing Leibniz's rule are:

$$\frac{\partial}{\partial t_a} f_1(t) = \Delta a_{da} \quad \frac{\partial}{\partial d} f_1(t) = a_d \tag{15}$$

The partial derivatives of the HIC function with respect to t_a and d can therefore be written as:

$$\frac{\partial HIC(t_a, d)}{\partial t_a} = 2.5 d^{-1.5} \Delta a_{da} \left(f_1(t)\right)^{1.5} \tag{16}$$

$$\frac{\partial HIC(t_a,d)}{\partial d}=1.5d^{-1.5}\left(f_1(t)\right)^{1.5}\left(\frac{5}{3}a_d-d^{-1}\left(f_1(t)\right)\right) \quad (17)$$

The four second partial derivatives of the HIC function are:

$$\frac{\partial^2 HIC(t_a,d)}{\partial t_a^{\,2}}=\frac{5}{2}d^{-1.5}\left(f_1(t)\right)^{0.5}\left(\frac{\partial \Delta a_{da}}{\partial t_a}f_1(t)+\frac{3}{2}\Delta a_{da}^{\,2}\right) \quad (18)$$

$$\frac{\partial^2 HIC(t_a,d)}{\partial t_a\partial d}=\frac{5}{2}d^{-1.5}\left(f_1(t)\right)^{0.5}\left(\frac{\partial \Delta a_{da}}{\partial d}f_1(t)+\frac{3}{2}\Delta a_{da}\left(a_d-d^{-1}f_1(t)\right)\right) \quad (19)$$

$$\frac{\partial^2 HIC(t_a,d)}{\partial d\partial t_a}=\frac{3}{2}d^{-1.5}\left(f_1(t)\right)^{0.5}\left(\frac{5}{3}\frac{\partial a_d}{\partial t_a}f_1(t)+\frac{5}{2}\Delta a_{da}\left(a_d-d^{-1}f_1(t)\right)\right) \quad (20)$$

$$\frac{\partial^2 HIC(t_a,d)}{\partial d^2}=\frac{5}{2}d^{-1.5}\left(f_1(t)\right)^{0.5}\left(\frac{\partial a_d}{\partial d}f_1(t)+\frac{3}{2}\left(a_d-d^{-1}f_1(t)\right)^2\right) \quad (21)$$

When N ≥ 1, there exist N – 1 temporal domain delimited acceleration functions. The evaluation of KKT points within each acceleration function, inclusive of the boundary points of the same, involves the first and second order partial derivatives of the forms presented above. Additionally, as noted previously, a check is required across each internal transition point. The evaluation across the k^{th} internal boundary point {k: 1 ≤ k ≤ N – 1} requires a check across the k^{th} and $k+1^{st}$ acceleration functions. The former is denoted as $a_k(t)$ and the latter is denoted as $a_m(t)$. The HIC function can then be defined as:

$$HIC(t_a,d)=d^{-1.5}\left(\int_{t_a}^{t_k}a_k(t)dt+\int_{t_k}^{t_a+d}a_m(t)dt\right)^{2.5} \quad (22)$$

The following notation is used to simplify the presentation: $a_k(t_a) = a_{ka}$, $a_m(t_a + d) = a_{md}$ and $a_{md} - a_{ka} = \Delta a_{mdka}$. Following the form of equation (14), the presentation may be simplified by the introduction of:

$$\int_{t_a}^{t_k}a_k(t)dt+\int_{t_k}^{t_a+d}a_m(t)dt=f_{km}(t) \quad (23)$$

The partial derivatives of $f_{km}(t)$ with respect to t_a and with respect to d can therefore be written as:

$$\frac{\partial}{\partial t_a}\left(f_{km}(t)\right)=\Delta a_{mdka} \qquad \frac{\partial}{\partial d}\left(f_{km}(t)\right)=a_{md} \quad (24)$$

The first partial derivatives of the HIC function for this case are:

$$\frac{\partial HIC(t_a,d)}{\partial t_a}=\frac{5}{2}\Delta a_{mdka}d^{-1.5}\left(f_{km}(t)\right)^{1.5} \quad (25)$$

$$\frac{\partial HIC(t_a,d)}{\partial d}=d^{-1.5}\left(f_{km}(t)\right)^{1.5}\left(\frac{5}{2}a_{md}-\frac{3}{2}d^{-1}f_{km}(t)\right) \quad (26)$$

The second partial derivatives of the HIC function this case are:

$$\frac{\partial^2 HIC(t_a,d)}{\partial t_a^{\,2}}=\frac{5}{2}d^{-1.5}\left(f_{km}(t)\right)^{0.5}\left(\frac{3}{2}\Delta a_{mdka}^{\,2}+\frac{\partial \Delta a_{mdka}}{\partial t_a}f_{km}(t)\right) \quad (27)$$

$$\frac{\partial^2 HIC(t_a,d)}{\partial t_a\partial d}=\frac{5}{2}d^{-1.5}\left(f_{km}(t)\right)^{0.5}\cdot\left(\frac{\partial \Delta a_{mdka}}{\partial d}f_{km}(t)+\frac{3}{2}\Delta a_{mdka}\left(a_{md}-d^{-1}\left(f_{km}(t)\right)^{0.5}\right)\right) \quad (28)$$

$$\frac{\partial HIC(t_a,d)}{\partial d\partial t_a}=\frac{5}{2}d^{-1.5}\left(f_{km}(t)\right)^{0.5}\cdot\left(\frac{\partial a_{md}}{\partial t_a}f_{km}(t)+\frac{3}{2}\Delta a_{mdka}\left(a_{md}-d^{-1}f_{km}(t)\right)\right) \quad (29)$$

$$\frac{\partial^2 HIC(t_a,d)}{\partial d^2}=\frac{5}{2}d^{-1.5}\left(f_{km}(t)\right)^{0.5}\left(\frac{\partial a_{md}}{\partial d}f_{km}(t)+\frac{3}{2}\left(a_{md}-d^{-1}f_{km}(t)\right)^2\right) \quad (30)$$

2.4 First order KKT conditions for the HIC

For the case in which a(t) is defined by a single analytic function, the Lagrangian function as per equation (2) is:

$$L(t_a,d,\boldsymbol{\mu},\mathbf{s}) = HIC(t_a,d) + \sum_{j=1}^{3}\mu_j\left(g_j(t_a,d)+s_j^2\right) \quad (31)$$

The coupled vector equations for the first order KKT conditions are:

$$\frac{\partial L(t_a,d,\boldsymbol{\mu},\mathbf{s})}{\partial t_a} = 2.5d^{-1.5}\Delta a_{da}\left(f_1(t)\right)^{1.5} + \mu_2 - \mu_3 = 0 \quad (32)$$

$$\frac{\partial L(t_a,d,\boldsymbol{\mu},\mathbf{s})}{\partial d} = 1.5d^{-1.5}\left(f_1(t)\right)^{1.5}\cdot \left(\frac{5}{3}a_d - d^{-1}\left(f_1(t)\right)\right) - \mu_1 - \mu_3 = 0 \quad (33)$$

$$\begin{aligned} d_{max} - d + s_1^2 &= 0 \\ t_a - t_0 + s_2^2 &= 0 \\ t_N - t_a - d + s_3^2 &= 0 \end{aligned} \quad (34)$$

$$2\mu_1 s_1 = 0 \quad 2\mu_2 s_2 = 0 \quad 2\mu_3 s_3 = 0 \quad (35)$$

For the case in which a(t) is comprised of N-1 {N: N > 1} temporal domain delimited acceleration functions, the first order KKT equations for each function involves evaluating equations (32-35) as written for the k^{th} acceleration function. In this regard, the term k is added to each subscript in the definitions and equations presented for the case of a = a(t). For the evaluation across the k^{th} transition point, the second and third constraint conditions can be dropped. The resulting Lagrangian is:

$$L(t_a,d,\boldsymbol{\mu},\mathbf{s}) = HIC(t_a,d) + \mu_1\left(g_1(t_a,d)+s_1^2\right) \quad (36)$$

The four coupled equations for the first order KKT conditions are:

$$\frac{\partial L(t_a,d,\boldsymbol{\mu},\mathbf{s})}{\partial t_a} = 2.5\Delta a_{mdka}d^{-1.5}\left(f_{km}(t)\right)^{1.5} = 0 \quad (37)$$

$$\frac{\partial L(t_a,d,\boldsymbol{\mu},\mathbf{s})}{\partial d} = 1.5d^{-1.5}\left(f_{km}(t)\right)^{1.5}\cdot \left(\frac{5}{3}a_{md} - d^{-1}f_{km}(t)\right) - \mu_1 = 0 \quad (38)$$

$$d_{max} - d + s_1^2 = 0 \quad (39)$$

$$2\mu_1 s_1 = 0 \quad (40)$$

2.5 Second order KKT conditions for the HIC

The Hessian in equation (8), for the HIC function, is of the form:

$$\nabla^2 HIC(\tau_a,\delta) = \begin{bmatrix} \dfrac{\partial^2 HIC(\tau_a,\delta)}{\partial t_a^2} & \dfrac{\partial^2 HIC(\tau_a,\delta)}{\partial t_a \partial d} \\ \dfrac{\partial^2 HIC(\tau_a,\delta)}{\partial d \partial t_a} & \dfrac{\partial^2 HIC(\tau_a,\delta)}{\partial d^2} \end{bmatrix} \quad (41)$$

Where τ_a and δ are the KKT point values of t_a and d, respectively. The first order gradients of the inequality constraints are:

$$\nabla g_1(\tau_a,\delta) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \nabla g_2(\tau_a,\delta) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \nabla g_3(\tau_a,\delta) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad (42)$$

The Hessian for each of the inequality constraints is $[0]_{2x2}$. The second order gradients of the inequality constraints are all zero valued. Equation (8), therefore, reduces to:

$$\mathbf{z}^T\nabla^2 HIC(\tau_a,\delta)\mathbf{z} > 0 \quad \forall \mathbf{z} \in \mathbb{R}^n, \mathbf{z} \neq \mathbf{0} \quad (43)$$

Expanding this equation leads to the following result:

$$z_1^2\frac{\partial^2 HIC(\tau_a,\delta)}{\partial t_a^2} + z_1 z_2\left(\frac{\partial^2 HIC(\tau_a,\delta)}{\partial d \partial t_a} + \frac{\partial^2 HIC(\tau_a,\delta)}{\partial t_a \partial d}\right) + z_2^2\frac{\partial^2 HIC(\tau_a,\delta)}{\partial d^2} > 0 \quad (44)$$

When only the first inequality constraint is active, $\mathbf{z} = \{z_1, 0\}^T$. When only the second inequality constraint is active, $\mathbf{z} = \{0, z_2\}^T$. The first case results in the reduction of equation (44) solely to the quadratic term in z_1^2. The second case results in the reduction of equation (44) solely to the quadratic term in z_2^2. When only the third inequality is active, $\mathbf{z} = \{z_1, -z_1\}$. For the case in which the HIC function is based upon a single acceleration function, equation (44) becomes:

$$\frac{5}{2}d^{-1.5}\left(f_1(t)\right)^{0.5}\left(\begin{matrix} f_1(t)\left(\begin{matrix}\left(\frac{\partial \Delta a_{da}}{\partial t_a}\right)z_1^{\,2}+ \\ \left(\frac{\partial \Delta a_{da}}{\partial d}+\frac{\partial a_d}{\partial t_a}\right)z_1 z_2 \\ +\frac{\partial a_d}{\partial d}z_2^{\,2}\end{matrix}\right)+ \\ \frac{3}{2}\left(\begin{matrix}\Delta a_{da}z_1+ \\ \left(a_d-d^{-1}f_1(t)\right)z_2\end{matrix}\right)^2\end{matrix}\right)>0 \quad (45)$$

For the case in which the HIC is a function of two resultant acceleration functions, equation (44) becomes:

$$\frac{5}{2}d^{-1.5}\left(f_{km}(t)\right)^{0.5}\left(\begin{matrix} f_{km}(t)\left(\begin{matrix}\frac{\partial \Delta a_{mdka}}{\partial t_a}z_1^{\,2}+ \\ \left(\begin{matrix}\frac{\partial \Delta a_{mdka}}{\partial d}+ \\ \frac{\partial a_{md}}{\partial t_a}\end{matrix}\right)z_1 z_2+ \\ \frac{\partial a_{md}}{\partial d}z_2^{\,2}\end{matrix}\right)+ \\ \frac{3}{2}\left(\begin{matrix}\Delta a_{mdka}z_1+ \\ \left(a_{md}-d^{-1}f_{km}(t)\right)z_2\end{matrix}\right)^2\end{matrix}\right)>0 \quad (46)$$

In both of these equations, the term that multiplies the parenthetical is positive. Inside of each parenthetical, the second additive term is positive given that it is a squared term. When either z_1 (second inequality constraint as the sole active constraint) or z_2 (first inequality constraint as the sole active constraint) are zero valued, the first additive term in the parenthetical is positive for any real non-zero valued z_2 or z_1, respectively.

3. APPLICATIONS

3.1 HIC based on singular a(t)

The coupled vector equations found under equations (32-35) are the operative first order KKT equations for the case when the HIC function is based upon a single acceleration function a(t). As noted previously, there are eight cases that are generated by the two unknowns and the three inequality constraints. These eight cases can be evaluated for any singular function a(t) without further specifying the form of a(t) in an a priori manner. These cases along with the solutions generated from the switching conditions, are listed in Table 1.

Table -1: Eight constrained cases for singular a(t)

Case	Active $g_j(t_a, d)$			Solutions from s_j^2	
	1	2	3	t_a	d
1					
2	X				d_{max}
3		X		t_0	
4			X		t_N - t_a
5	X	X		t_0	d_{max}
6	X		X	t_N - d_{max}	d_{max}
7		X	X	t_0	$t_N - t_0$
8	X	X	X	t_0	d_{max}

The second case, in Table 1, is of specific interest. The detailed derivation for this case is presented below and the results for all eight cases are shown in Table 2. For the second case, in which only the first constraint is active, s_1^2 = 0, μ_2 = 0 and μ_3 = 0. From the first equation under (34), d = d_{max}. The second and third equations under (34) can be directly solved, respectively, for s_2^2 and s_3^2.

$$s_2^{\,2}=t_0-t_a \quad s_3^{\,2}=t_a+d_{max}-t_N \quad (47)$$

A valid solution for this problem is one for which $s_2^2 \le 0$ and $s_3^2 \le 0$. The remaining unknowns are t_a and μ_1. Substitution of $\mu_2 = \mu_3 = 0$ and d = d_{max} into equation (33) and solving for μ_1 leads to the following solution.

$$\mu_1=1.5d_{max}^{\;-1.5}\left(f_1(t)\right)^{1.5}\left(\frac{5}{3}a_d-d_{max}^{\;-1}\left(f_1(t)\right)\right) \quad (48)$$

Because d_{max} is implicitly greater than zero and the integral that defines $f_1(t)$ is greater than zero, the parenthetical must be non-negative. Such a case occurs when the following holds:

$$a_d \ge \frac{3}{5}d_{max}^{\;-1}\int_{t_a}^{t_a+d_{max}} a(t)dt \quad (49)$$

For these equations, a_d is equal to $a(t_a + d_{max})$. Substitution of $\mu_2 = \mu_3 = 0$ and d = d_{max} into equation (32) leads to the following result:

$$2.5d^{-1.5}\Delta a_{da}\left(f_1(t)\right)^{1.5}=0 \quad (50)$$

Because d and $f_1(t)$ are non-zero valued, solution to this equation requires Δa_{da} to be zero valued.

$$a\left(t_a+d_{max}\right)=a_d=a_a=a\left(t_a\right) \quad (51)$$

Therefore, for this case, the solution for t_a is that which results in a_a being equal to a_d and for which either value is greater than or equal to three fifths of the average acceleration between a_a and a_d.

Table -2: Solutions from μ_i for singular a(t)

Case	Solutions and requirements from μ_i
1	$a_d = a_a$ and $a_a = \frac{3}{5} d^{-1} \int_{t_a}^{t_a+d} a(t) dt$
2	$a_d = a_a$ and $a_d \geq \frac{3}{5} d_{max}^{-1} \int_{t_a}^{t_a+d_{max}} a(t) dt$
3	$a_d \leq a_a$ and $a_d = \frac{3}{5} d^{-1} \int_{t_0}^{t_0+d} a(t) dt$
4	$a_d = a_N$ and $a_a = \frac{3}{5} (t_N - t_a)^{-1} \int_{t_a}^{t_N} a(t) dt$
5	$a_d \leq a_0$ and $a_d \geq \frac{3}{5} d_{max}^{-1} \int_{t_0}^{t_0+d_{max}} a(t) dt$
6	$a_d \geq a_a$ and $a_a \geq \frac{3}{5} d_{max}^{-1} \int_{t_N-d_{max}}^{t_N} a(t) dt$
7	$a_a \geq \frac{3}{5} (t_N - t_0)^{-1} \int_{t_0}^{t_N} a(t) dt$ and $a_d \geq \frac{3}{5} (t_N - t_0)^{-1} \int_{t_0}^{t_N} a(t) dt$
8	$d_{max} = t_N - t_0$

3.2 HIC based on multiple a(t)

This section focuses on the evaluation of those situations in which the form of the underlying acceleration functions changes at an internal transition point and specifically on the evaluation of the case where t_a is within the domain of $a_k(t)$ and $t_a + d$ is in the domain of $a_m(t)$ where $m = k + 1$. Generally, this evaluation is required for all internal transition points k and m. As noted before, these evaluations are in addition to the evaluations that are done within the domain of each function. Only the first constraint is considered, herein, as potentially being active. When this constraint is inactive, $\mu_1 = 0$ and by equation (39), $s_1^2 = d - d_{max}$. A valid solution is one where $s_1^2 \leq 0$. From equation (37), for the equation to be zero-valued, the term Δa_{mdka} must be zero valued. Therefore:

$$\Delta a_{mdka} = 0 \rightarrow a_m (t_a + d_{max}) = a_k (t_a) \quad (52)$$

Because μ_1 is equal to zero, the equating of equations (37) and (38) results in the following:

$$d^{-1.5} (f_{km}(t))^{1.5} \left(\frac{5}{2} a_{ka} - \frac{3}{2} d^{-1} f_{km}(t) \right) = 0 \quad (53)$$

For this equation to hold, the parenthetical must be zero-valued. This results in the following solution.

$$a_k (t_a) = \frac{3}{5} d^{-1} \left(\int_{t_a}^{t_k} a_k(t) dt + \int_{t_k}^{t_a+d} a_m(t) dt \right) \quad (54)$$

Therefore, for the unconstrained case, the solution that maximizes the HIC function across the internal transition point, t_k, occurs at the values of $\{t_a, d\}$ where the acceleration $a_k(t_a)$ is equal to $a_m(t_a + d)$ and where the acceleration at each point is equal to three fifths of the average acceleration between the two points. This was the solution derived previously [12]. When the first constraint is active, $s_1^2 = 0$ and $d = d_{max}$. The requirement given by equation (52) remains valid for the constrained case. The solution for the Lagrange multiplier is obtained from equation (38).

$$\mu_1 = 1.5 d_{max}^{-1.5} (f_{km}(t))^{1.5} \left(\frac{5}{3} a_{md} - d_{max}^{-1} f_{km}(t) \right) \quad (55)$$

In order to meet the requirement that $\mu_1 \geq 0$, the parenthetical must be greater than or equal to zero. As a result:

$$a_m (t_a + d_{max}) \geq \frac{3}{5} d_{max}^{-1} \left(\int_{t_a}^{t_k} a_k(t) dt + \int_{t_k}^{t_a+d_{max}} a_m(t) dt \right) \quad (56)$$

Therefore, for the constrained case, the solution that maximizes the HIC function across the internal transition point, t_k, occurs at $d = d_{max}$, $a_k(t_a) = a_m(t_a + d_{max})$ and when both point acceleration values have a minimum value equal to three fifths of the average acceleration between the two points.

3.3 HIC based on specific singular a(t)

The simplest acceleration function that falls within this categorical descriptor is the ideal square wave, $a(t) = A_p$ where A_p is a constant value that is greater than zero over the domain $\{t: t_0 \leq t \leq t_1\}$. The average value of this function over any finite value of the operative domain is $0.5A_p$. This means that three fifths of the average acceleration over any finite value of the operative domain is $0.3A_p$. The point accelerations for $a_a = a_d = A_p$ for any two discrete points over the domain. From Table 1, cases 1, 3 and 4 are excluded based on form. The remaining cases, however, cover any value of t_a limited to $\{t_a: t_0 \leq t \leq t_1 - d\}$. This is the expected result.

The next level of complexity for this case are those functions that are monotonic. For a function of time, the function is monotonic increasing over its domain if the sign of the first time derivative is positive over the domain.

Similarly, for a function of time, a function is monotonic decreasing over its domain if the sign of the first time derivative is negative over the domain. The solutions for both cases are obvious by informal inspection. For a monotonic increasing function, $t_a = t_N - d_{max}$ and $d = d_{max}$. For a monotonic decreasing function, $t_a = t_0$ and $d = d_{max}$. Formally, for the family of monotonic increasing functions, cases 1-3 and 5 can readily be excluded based on the relationship between a_a and a_d. For the family of monotonic decreasing functions, cases 1-2, 4 and 6 can be excluded for the same reason. Case 8 is valid for monotonic increasing and decreasing functions when $d_{max} = t_1 - t_0$.

Finally, one may consider the situation in which a(t) is modeled as a symmetric half sine pulse shape. There are three transition points that are of relevance for this function. The first and last, $t = t_0$ and $t = t_2$, respectively denote the time at which a(t) initiates and terminates. The second transition point, $t = t_1$, denotes both the time at which a(t) reaches its maximum value (i.e. $t = t_1 = t_m$) and denotes the time value about which a(t) exhibits symmetry. Because of this symmetry, for this formulation, $\Delta t_{10} = \Delta t_{21} = 0.5\Delta t_{20}$ and $a_0 = a_2$. If T denotes the full period of the sinusoid, then $\Delta t_{20} = 0.5T$. The form of the acceleration function is:

$$\begin{aligned} a(t) &= a_0 + \Delta a_{10} \sin\left(\frac{2\pi}{T}(t - t_0)\right) \\ &= a_0 + \Delta a_{10} \sin\left(\frac{\pi}{\Delta t_{20}}(t - t_0)\right) \end{aligned} \tag{57}$$

The case in which only the first constraint is active (i.e. case 2 of Table 1) is the sole case of interest. This particular case requires that $a_a = a_d$, which for this specific function means:

$$\begin{aligned} &a_0 + \Delta a_{10} \sin\left(\frac{2\pi}{T}(t_a - t_0)\right) = \\ &a_0 + \Delta a_{10} \sin\left(\frac{2\pi}{T}(t_a + d_{max} - t_0)\right) \end{aligned} \tag{58}$$

Subtracting a_0 from both sides of the equality and dividing both sides by Δa_{10} leads to the following result:

$$\sin\left(\frac{2\pi}{T}(t_a - t_0)\right) = \sin\left(\frac{2\pi}{T}(t_a + d_{max} - t_0)\right) \tag{59}$$

Because of the periodic nature of the sine function, the solution for t_a is of the form:

$$t_a = t_0 \pm \frac{T}{4} + c_1 - \frac{d_{max}}{2} \quad c_1 \in \mathbb{Z} \tag{60}$$

Letting $c_1 = 0$, using the (+) form of the quarter period and substituting $T = 0.25\Delta t_{10}$ followed by simplification leads to the following, expected, solution.

$$t_a = t_1 - \frac{d_{max}}{2} \tag{61}$$

At $t = t_d = t_1 + 0.5d_{max}$, a(t) is:

$$a_d = a_0 + \Delta a_{10} \cos\left(\frac{\pi d_{max}}{4\Delta t_{10}}\right) \tag{62}$$

Three fifths of the average acceleration between the two points is:

$$\frac{3}{5} d_{max}^{-1} \int a(t)dt = \frac{3}{5} a_0 + \frac{12\Delta t_{10}\Delta a_{10}}{5\pi d_{max}} \sin\left(\frac{\pi d_{max}}{4\Delta t_{10}}\right) \tag{63}$$

For the sign of the Lagrange multiplier to be correct, thereby producing a valid KKT point, the following must hold:

$$\begin{aligned} &2\frac{a_0}{\Delta a_{10}}\left(\frac{\pi d_{max}}{4\Delta t_{10}}\right) \geq \\ &3\sin\left(\frac{\pi d_{max}}{4\Delta t_{10}}\right) - 5\left(\frac{\pi d_{max}}{4\Delta t_{10}}\right)\cos\left(\frac{\pi d_{max}}{4\Delta t_{10}}\right) \end{aligned} \tag{64}$$

The ratio of $d_{max}/\Delta t_{10}$ will be valued between 0 and 1. At the limit of 0, equation (64) reduces to 0 = 0. At the limit of 1, the right side the equation reduces to $2^{-0.5}(3 - 1.25\pi) \sim -0.655481$. The left side of the equation, however, is positive valued and the inequality holds. The result regarding the location of t_a and t_d do not change for the case of the exponentiated sine function (e.g. haversine or $\sin^2$ or for any $\sin^p$ where p is real-valued and greater than or equal to unity) whereas the form of equations (63-64) will change.

3.4 HIC based on specific multiple a(t)

The simplest case where multiple acceleration functions are apt is the triangular pulse shape. The temporal domain of this function contains three transition points with $t = t_1$ denoting the point of C^0 continuity between the first linear function, $a_1(t)$, and the second linear function, $a_2(t)$. The acceleration functions are:

$$\begin{aligned} a_1(t) &= a_0 + k_1(t - t_0) \quad \{t : t_0 \leq t \leq t_1\} \\ a_2(t) &= a_1 + k_2(t - t_1) \quad \{t : t_1 \leq t \leq t_2\} \end{aligned} \tag{65}$$

The slope of the first linear region is positive while the slope of the second liner region is negative. Both slopes can be written in terms of the boundary values for each function.

$$k_1 = \frac{\Delta a_{10}}{\Delta t_{10}} \quad k_2 = \frac{\Delta a_{21}}{\Delta t_{21}} \tag{66}$$

The triangular pulse is fully symmetric when $k_1 = -k_2$ and $\Delta t_{10} = \Delta t_{21}$. The evaluation, limited to an interval that spans t

$= t_1$, for $d = d_{max}$, requires determining t_a for $a_1(t_a) = a_2(t_a + d_{max})$.

$$t_a = \frac{\Delta a_{10} + k_1 t_0 + k_2\left(d_{max} - t_1\right)}{k_1 - k_2} = t_1 + \frac{\Delta a_{12}\Delta t_{01}}{a_0\Delta t_{12} + a_1\Delta t_{20} + a_2\Delta t_{01}} d_{max} \quad (67)$$

For the Lagrange multiplier as per equation (56):

$$a_2\left(t_d\right) = a_{2d} = a_1 - \frac{\Delta a_{10}\Delta a_{12}}{\Delta a_{10}\Delta t_{21} + \Delta a_{12}\Delta t_{10}} d_{max} \quad (68)$$

$$\frac{3}{5} d_{max}{}^{-1}\int a(t)dt = \frac{3}{5}a_1 - \frac{3}{10}\frac{\Delta a_{10}\Delta a_{12}}{\Delta a_{10}\Delta t_{21} + \Delta a_{12}\Delta t_{10}} d_{max} \quad (69)$$

Multiple formulations exist by which the rise phase function, $a_1(t)$, and the fall phase function, $a_2(t)$, can both be modeled using trigonometric functions. The simplest formulation is for the case in which both phases are quarter period responses such that $\Delta t_{10} = 0.25T_1$ and $\Delta t_{21} = 0.25T_2$. The operative equations for such a formulation are:

$$a_1\left(t\right) = a_0 + \Delta a_{10}\sin\left(\frac{\pi}{2\Delta t_{10}}\left(t - t_0\right)\right) \quad \left\{t : t_0 \le t \le t_1\right\} \quad (70)$$

$$a_2\left(t\right) = a_2 + \Delta a_{12}\cos\left(\frac{\pi}{2\Delta t_{21}}\left(t - t_1\right)\right) \quad \left\{t : t_1 \le t \le t_2\right\} \quad (71)$$

When $t = t_1$, the operand of the sine function in equation (70) becomes 0.5π, which leads to the maximum positive value of the sine function of unity. The result is that $a_1(t_1) = a_0 + (a_1 - a_0) = a_1$. When $t = t_1$, the operand of the cosine function in equation (71) becomes 0, which leads to the maximum positive value of the cosine function of unity. The result is that $a_2(t_1) = a_2 + (a_1 - a_2) = a_1$. Therefore, for this case, the time at which the maximum of the resultant acceleration occurs is $t = t_m = t_1$. Because each segment is a quarter phase response, $a_0 = a_2$. The requirement that $a_1(t_a) = a_2(t_a + d_{max})$, for the second case in Table 1, leads to the following:

$$a_0 + \Delta a_{10}\sin\left(\frac{\pi}{2\Delta t_{10}}\left(t_a - t_0\right)\right) = a_2 + \Delta a_{12}\cos\left(\frac{\pi}{2\Delta t_{21}}\left(t_a + d_{max} - t_1\right)\right) \quad (72)$$

Because $a_0 = a_2$, $\Delta a_{10} = \Delta a_{12}$. This leads to the following solution:

$$t_a = t_1 - d_{max}\frac{\Delta t_{10}}{\Delta t_{20}} \quad (73)$$

This solution reduces to the expected result of $t_a = t_1 - 0.5d_{max}$ when $\Delta t_{10} = 0.5\Delta t_{20}$. Based on the solution shown in equation (73):

$$t_d = t_a + d_{max} = t_1 + d_{max}\left(1 - \frac{\Delta t_{10}}{\Delta t_{20}}\right) \quad (74)$$

If $\alpha = \Delta t_{10}/\Delta t_{20}$, where α such that $\{\alpha: 0 < \alpha < 1\}$, then $\Delta t_{21}/\Delta t_{20} = 1 - \alpha$. The requirement for the Lagrange multiplier, given by equation (56), becomes:

$$\frac{\pi}{3}\frac{a_0}{\Delta a_{10}} + \frac{5\pi}{6}\cos\left(\frac{\pi d_{max}}{2\Delta t_{20}}\right) - \frac{\Delta t_{20}}{d_{max}}\sin\left(\frac{\pi d_{max}}{2\Delta t_{20}}\right) \ge 0 \quad (75)$$

The ratio of $d_{max}/\Delta t_{20}$ is such that $0 < d_{max}/\Delta t_{20} \le 1$. The limit of the summation of the last two terms, as the ratio approaches zero is $\pi/3$. The summation of the last two terms is zero valued when the ratio is approximately $2.10558\pi^{-1}$. When the ratio is unity, the summation of the last two terms is -1. It can therefore be stated that the constraint on the Lagrange multiplier is conditionally met. However, for most realistic applications, the ratio of $d_{max}/\Delta t_{20}$ is well below unity value and the requirements of the constraint will be met.

The final case for consideration is that of situation in which a(t) is modeled as a trapezoidal function, which in turn is a special case of the use of three domain delimited linear functions. The trapezoidal function is characterized by a linear rise phase, a constant valued plateau and a linear fall phase. The four transition points result in three temporal domains consisting of $\{t: t_0 \le t \le t_1\}$, $\{t: t_1 \le t \le t_2\}$ and $\{t: t_2 \le t \le t_3\}$. The acceleration functions corresponding to each temporal domain are $a(t) = a_0 + k_1 (t - t_0)$, $a(t) = a_1 = a_2$ and $a(t) = a_2 + k_2 (t - t_2)$. When $\Delta t_{21} = d_{max}$, then $t_a = t_1$ and $t_a + d_{max} = t_2$. When $\Delta t_{21} > d_{max}$ then $t_1 \le t_a \le t_2 - d_{max}$. When $\Delta t_{21} < d_{max}$, the relevant checks are of the acceleration function across t_1, the acceleration function across both t_1 and t_2 and the acceleration function across t_2.

$$HIC\left(t_a, d\right) = d^{-1.5}\left(\int_{t_a}^{t_1}\left(a_0 + k_1\left(t - t_0\right)\right)dt + \int_{t_1}^{t_1+d} a_1 dt\right)^{2.5} \quad (76)$$

$$\mathrm{HIC}(t_a, d) = d^{-1.5}\left(\begin{array}{l}\int_{t_a}^{t_1}\left(a_0 + k_1\left(t - t_0\right)\right)dt + \\ \int_{t_1}^{t_2} a_1 dt + \\ \int_{t_2}^{t_a+d}\left(a_1 + k_2\left(t - t_2\right)\right)dt\end{array}\right)^{2.5} \quad (77)$$

$$\mathrm{HIC}(t_a, d) = d^{-1.5}\left(\int_{t_a}^{t_2} a_1 dt + \int_{t_2}^{t_a+d}\left(a_1 + k_2\left(t - t_2\right)\right)dt\right)^{2.5} \quad (78)$$

The solution for t_a for equation (76) and equation (78) can be determined based upon equation (52). For the case of equation (76), $t_a = t_1$. However, when $\Delta t_{21} < d_{max}$ then $t_2 = t_a + d_{max}$ and $t_a = \Delta t_{21} - d_{max}$. The solution for t_a for equation (78) is $t_a = t_2 - d$. However, when $\Delta t_{21} < d_{max}$, $t_a = t_1$. For the case of equation (77), with $d = d_{max}$, the solution for $t = t_a$ is:

$$t_a = \frac{\Delta a_{10} + k_1 t_0 + k_2\left(d_{max} - t_2\right)}{\Delta k_{12}} \quad (79)$$

The solution for t_a is more heavily weighted towards the linear function that has the smaller slope magnitude.

4. DISCUSSION

The quantification of the maximum HIC value, based on anthropometric test device head center of mass resultant acceleration calculations from triaxial translational acceleration measurements, serves as a key metric when it comes to passing regulatory compliance requirements for the sale of production motor vehicles to the general public in the United States. The quantification of the same metric also serves as an important predictor, in a forensic sense, when it comes to predicting the potential for closed head injury when based upon a base excitation (i.e. vehicular collision partner collision severity from an accident reconstruction evaluation) coupled with an occupant kinematics analysis. From the mathematical perspective, the importance of constraining the temporal duration of the calculation can readily be seen by considering the HIC equation when a(t) is a 1 G square wave and when the duration is 1 second. The result is a HIC value of 1000. The problematic nature of HIC calculations for which the temporal duration is unconstrained is the same problem that was present for the SI. HIC calculations, correspondingly, are conducted with a maximum limit on the temporal window length. This limit is generally either 15 milliseconds or 36 milliseconds [14]. Injury assessment reference values are typically presented in conjunction with the length of the temporal window. For example, a HIC_{15} value of 700 represents the IARV for the HIC function, with a maximal temporal window length of 15 milliseconds, and which represents a five percent probability of a severe head injury with severe being coded as a 4+ score on the Abbreviated Injury Scale [15]. For the case of discrete resultant acceleration data, the unoptimized approach would consist of calculating the HIC value over the interval $\{t_i, t_i + n\Delta t\}$,where Δt is the sampling interval and n $\{n: 1 \le n \le d_{max}/\Delta t, n \in \mathbb{Z}\}$.

The computational requirements for the case of discrete resultant acceleration data are substantively mitigated when the acceleration is known or modeled as an analytical closed form function. The findings of previous published research work [12] are seminal in regards to showing that the HIC function is maximized, for the unconstrained case, over the interval $t = t_1 = \tau_1$ and $t = t_2 = \tau_2$, when $a(\tau_1) = a(\tau_2)$ and when the magnitude of the resultant acceleration at either point is equal to three fifths of the average acceleration between the points. Functionally, the evaluation of the HIC function at the boundary points of $t = t_0$ and $t = t_N$ is requisite, in the general sense, when $a(t_0)$ falls within the domain of monotonically decreasing function or when $a(t_N)$ falls within the domain of a monotonically increasing function. Excluding the square wave and the trapezoidal acceleration cases, the other acceleration functions of interest are readily characterizable as being concave. The determination of concavity can readily be made in a priori manner due to the fact that the acceleration functions are specified. When concavity is an apt descriptor of the underlying resultant acceleration function, the determination of the domain that maximizes the HIC function reduces to the evaluation of points internal to the temporal domain of the resultant acceleration function. When the temporal domain of the resultant acceleration function exceeds a maximum window or clipping length, the optimization problem, in this case being a maximization problem, reduces to the evaluation of case 2 as per Table 1. For this case, there is a single switching condition and a single Lagrange multiplier. The former encodes whether or not the constraint is active while the latter encodes the inequality constraint of $d_{max} - d \ge 0$, as an additive term to the HIC function, thereby generating the Lagrangian for the problem. The Lagrangian, in turn, represents the converted form of the constrained problem into one for which the first derivative test of the unconstrained problem still applies. Thusly, it comes as no surprise that the finding of $a(\tau_a) = a(\tau_a + d_{max})$ still holds when the sole constraint for the case is active (the first partial derivative of the Lagrangian with respect to t_a sees no contribution from the constraint). The evaluation of the problem with the constraint as active, which represents the objective of the subject work, does lead to the following additional results. The first is the expected result of $d = d_{max}$. The second is that the equally valued resultant acceleration at $t = \tau_a$ and $t = \tau_a + d_{max}$ is greater than or equal to three fifths of the average acceleration between the points. These solutions hold for concave resultant acceleration functions, irrespective of the complexity of the same (implicit in this statement that is that there is a one-to-one correspondence between any value of t within the domain and a(t)).

The specific forms of the resultant acceleration function, both as singular functions and as composite temporally domain limited functions, were chosen primarily because of commonality of usage. Clearly, the set of functions presented is a subset of the totality of functions that could be considered. Furthermore, for the composite case, one may readily use functions from different families (e.g. a quarter sine initial phase followed by a subsequent linear phase). With the first inequality constraint set to active, the requirement of equivalence in the magnitude of the resultant accelerations for each function pair that generates an internal transition point remains. Furthermore, the requirement imposed by the Lagrange multiplier remains. The specific form of this requirement varies as a function of the form of each equation that is involved in generating a transition point. For the case when only the first constraint is active, the requirements of the second order KKT are met for a valid first order KKT point.

5. CONCLUSIONS

The determination of the interval that maximizes the HIC function associated with an underlying resultant head acceleration, a(t), when the maximal length of the interval, itself, is subject to constraint, was evaluated using the standard first order necessary and second order sufficient KKT conditions for optimality. The mathematical framework enshrined within the KKT formulation for optimization provided a formal mechanism for evaluating the underlying the contextual problem of constrained maximization of the HIC function. For the case of internal points, with respect to the temporal domain of a(t), it was shown that when a(t) is a singular function, that the window length $d = d_{max}$, that the start of the maximal interval occurs at $a(t_a) = a(t_a + d_{max})$ and that the Lagrange multiplier has a valid sign when the resultant acceleration at either boundary of maximal interval is greater than or equal to three fifths of the average acceleration between and inclusive of the maximal interval temporal locations. For the case of internal points, with respect to the temporal domain of a(t), it was shown that when a(t) is a composite function, that the window length $d = d_{max}$, that the start of the maximal interval occurs at $a_k(t_a) = a_m(t_a + d_{max})$ and that the Lagrange multiplier has a valid sign when the resultant acceleration at either boundary of maximal interval is greater than or equal to three fifths of the average acceleration between and inclusive of the maximal interval temporal locations. These findings fill a longstanding gap in the research literature in regards to presenting the formally derived solutions for the constrained optimization of the HIC problem.

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BIOGRAPHY

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