A Study on Some Repairable Systems

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Abstract - In this paper, a new kind of series repairable system with repairman vacation is discussed, in which the failure rate functions of all the units and the delayed vacation rate function of the repairman are related to the working time of the system, and also considers a simple repairable system with a warning device which can signal an alarm when the system is not in good condition.

Key Words: Repairable system, delayed multiple vacations, semi linear evolution system, C_0-semi group theory, well-posedness, stability, sensitivity analysis

1. INTRODUCTION

A repairable system is a system which after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method rather than the replacement of the entire system. Since the 1960s, various repairable system models have been established and researched, such as one-unit, series, parallel, series-parallel, redundance, k-out-of-n, multiple-state, human machine and software systems. Vacation model originally arises in queueing theory and has been well studied in the past three decades and successfully applied in many areas such as manufacturing / service and computer / communication network systems.

In section 2, we discuss existence and uniqueness of system solution and some properties of the system is presented. In section 3, we discuss the reliability indexes. In section 4, we discuss the stability, especially the exponential stability of the system solution by C_0 semigroup theory.

2. Existence and Uniqueness of System Solution

The unique existence of the solution of the initial value problem of abstract semi-linear evolution equations. In this section, we discuss the unique existence of the mild solution of system because of its limitation of the physical condition, by C_0-semigroup theory. Some properties of the system operator A will be presented first.

Definition 2.1 The system operator $A : D(A) \subset X \to X$ defined by

$$A(P_{00}, P_{01}(x), P_{11}(x), \dots, P_{1n}(x), P_{21}(y), \dots, P_{2n}(y))^T$$

$$= \Bigg(\int_0^\infty r(x)\, P_{01}(x)\, dx + \sum_{i=1}^{n} \int_0^\infty \mu_i(y) P_{2i}(y) dy\,,$$

$$-P'_{01}(x) - r(x)P_{01}(x), -P'_{11}(x) - r(x)P_{11}(x)\,, \cdots,$$

$$-P'_{1n}(x) - r(x)P_{1n}(x), -P'_{21}(y) - \mu_1(y)P_{21}(y), \cdots,$$

$$-P'_{2n}(y) - \mu_n(y)P_{2n}(y)\Big)^T$$

with

$$D(A) = \begin{pmatrix} P = (P_{00}, P_{01}, P_{11}, \dots, P_{1n}, P_{21}, \dots, P_{2n})^T \in \mathrm{X} \\ P_j \text{ are differentiable in } \mathbb{R}_+ \text{and } P'_j \in L^1(\mathbb{R}_+), \\ j = 01, 1i, 2i, i = 1,2, \dots, n \end{pmatrix}.$$

Lemma 2.2 The system operator A is densely defined in X.

Proof. For any $F = (f_{00}, f_{01}, f_{11}, \dots, f_{1n}, f_{21}, \dots, f_{2n})^T \in X$, then $f_j \in L^1(\mathbb{R}_+)$, $j = 01, 1i, 2i$, $i = 1,2, \dots, n$. Thus for any $\eta > 0$, there exist positive numbers G_j and δ_j such that

$$\int_{G_{01}}^\infty |f_{01}(x)| dx < \frac{\eta}{9}, \qquad \int_0^{\delta_{01}} |f_{01}(x)| dx < \frac{\eta}{18}$$

and

$$\int_{G_k}^\infty |f_k(\xi)| d\xi < \frac{\eta}{9n}, \qquad \int_0^{\delta_k} |f_k(\xi)| d\xi < \frac{\eta}{18n},$$

$k = 1i, 2i, i = 1, \dots, n$. Let

$$\delta = \min\left\{\delta_{01}, \delta_{1i}, \delta_{2i}, \frac{3\eta}{2\{\eta r + 9[(\varepsilon + \Lambda)|f_{00}| + \int_0^\infty r(x)|f_{1i}(x)| dx]\}}\right\}.$$

Take $P_{00} = f_{00}$ and

$$P_{01}(x) = \begin{cases} \varepsilon P_{00}, & 0 \le x < \delta \\ g_{01}(x), & \delta \le x \le G_{01} \\ 0, & G_{01} < x < \infty \end{cases}$$

$$P_{1i}(x)=\begin{cases}0, & 0\le x<\delta\\ g_{1i}(x), & \delta\le x\le G_{1i}\\ 0, & G_{1i}<x<\infty\end{cases}$$

$$P_{2i}(y)=\begin{cases}\lambda_i P_{00}+\int_0^\infty r(x)P_{1i}(x)dx, & 0\le y<\delta\\ g_{2i}(y), & \delta\le y\le G_{2i}\\ 0, & G_{2i}<y<\infty.\end{cases}$$

Here, g_j are continuously differentiable functions satisfying

$$g_j(G_j)=0, g_{01}(\delta)=\varepsilon P_{00}, g_{1i}(\delta)=0,$$
$$g_{2i}(\delta)=\lambda_i P_{00}+\int_0^\infty r(x)P_{1i}(x)dx$$

and

$$\int_\delta^{G_{01}}|f_{01}(x)-P_{01}(x)|dx<\frac{\eta}{9}, \int_\delta^{G_k}|f_k(\xi)-P_k(\xi)|d\xi<\frac{\eta}{9n},$$

$j=01,k; k=1i,2i; i=1,2,\dots,n$. Then P_j are continuously differentiable functions and $P_j{}'\in L^1(\mathbb{R}_+)$. Thus $P=(P_{00},P_{01},P_{11},\dots,P_{1n},P_{21},\dots,P_{2n})^T\in D(A)$. Furthermore, it is easy to prove that $\| F-P \|<\eta$. Therefore, $D(A)$ is dense in X. ■

Lemma 2.3 $\{\xi|\xi>\varepsilon+\Lambda\}\subset\rho(A)$, where $\rho(A)$ is the resolvent set of system operator A. And there exists a constant $W>0$, such that for any $\xi>W$,

$$\| R(\xi;A)\| \le \frac{1}{\xi-W},$$

where $R(\xi;A)=(\xi I-A)^{-1}$.

Lemma 2.4 Let $f:[t_0,T]\times X\to X$ be continuous about t on $[t_0, T]$ and uniformly Lipschitz continuous (with constant L) on X, if $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t\ge 0$, on X, then for every $u_0\in X$ the initial value problem

$$\begin{cases}\frac{du(t)}{dt}+Au(t) & =f(t,u(t)), \quad t>t_0\\ \qquad u(t_0) & =u_0\end{cases}$$

has a unique mild solution $u\in C([t_0, T] : X)$. Moreover, the mapping $u_0\to u$ is Lipschitz continuous from X into $C([t_0, T] : X)$.

Theorem 2.5 The system operator A generates a C_0 semigroup $T(t)$.

Theorem 2.6 For any $T>0$, assume $\varepsilon(t)$ and $\lambda_i(t)$ $(i=1,2, \dots, n)$ are continuous on $[0, T]$. Then for any $P\in X$, if $P_{01}(t,\cdot)\in C([0,T]:L^1(\mathbb{R}_+))$, where P_{01} is the second component of P, the semi-linear evolution system has a unique mild solution $P\in C([0, T]:X)$. Moreover, the mapping $P_0\to P$ is Lipschitz continuous from X into $C([0, T] : X)$.

3. Reliability indexes

In this section, we substitute the limit values $\hat{\varepsilon}$ and $\hat{\lambda}_i$ respectively for the delayed vacation rate $\varepsilon(t)$ of repairman and the failure rate $\lambda_i(t)$ of each unit i, $i=1,2,\dots,n$. Thus system is stable. So in this section we can study steady-state reliability indexes of the system with the method of Laplace transformation because the premise of Laplace transformation needs the condition that the system solution, is unique existed and stable.

Theorem 3.1 The stead-state availability of the system is

$$A_v=\frac{1+\varepsilon f}{1+\varepsilon g+(1+\varepsilon f)\sum_{i=1}^n\lambda_i h_i}$$

where

$$f=\int_0^\infty e^{-\int_0^x[\Lambda+r(\tau)]d\tau}dx,$$
$$g=\int_0^\infty e^{-\int_0^x r(\tau)d\tau}dx,$$
$$h_i=\int_0^\infty e^{-\int_0^y \mu_i(\tau)d\tau}dy, \quad i=1,2,\dots,n.$$

Theorem 3.2 The stead-state probability of the repairman on vacation is

$$P_v=\frac{\varepsilon g}{1+\varepsilon g+(1+\varepsilon f)\sum_{i=1}^n\lambda_i h_i}.$$

Theorem 3.3 The stead-state failure frequency of the system is

$$W_f=\Lambda A_v.$$

4. Stability of System Solution

In this section, we discuss the stability, especially the exponential stability of the system solution by C_0 semigroup theory. For this purpose, we first translate the system equations into an abstract Cauchy problem in a suitable Banach space. Then some primary properties of system operator and its adjoint operator are presented.

4.1 Properties of System Operator A

In this section, we present some concerned properties of system operator A including the distribution of its spectrum.

Lemma 4.1 The system operator A is a densely closed dissipative operator.

Lemma 4.2 For any $\gamma \in \mathbb{C}$ satisfying $Re\gamma > 0$ or $\gamma = ia, a \in \mathbb{R}\backslash\{0\}$, γ is a regular point of the system operator A.

Lemma 4.3 0 is an eigenvalue of the system operator A with algebraic multiplicity one.

4.2. Properties of Adjoint Operator A^*

In this section, we present some properties of A^*, the adjoint operator of system operator A, including its spectrum distribution.

Lemma 4.4 For any $\gamma \in \mathbb{C}$ satisfying

$$\sup\left\{\frac{\varepsilon+\alpha_0}{|\gamma+\varepsilon+\alpha_0|}, \frac{\alpha_0+M}{Re\gamma+\alpha_0+M}, \frac{\lambda_1+\lambda_2}{|\gamma+\lambda_1+\lambda_2|}, \frac{\varepsilon}{|\gamma+\varepsilon|}, \frac{\lambda_1+\lambda_2+M}{Re\gamma+\lambda_1+\lambda_2+M}, \frac{M}{Re\gamma+M}\right\} < 1,$$

$\gamma \in \rho(A^*)$, the resolvent set of A^*, where $M = \sup\{\overline{r}, \overline{\mu}\}$ and $\overline{r}, \overline{\mu}$ are

$$\overline{r} = \sup_{x\in[0,\infty)} r(x) < \infty,$$

$$\overline{\mu} = \sup_{y\in[0,\infty)} \mu(y) < \infty.$$

Lemma 4.5 0 is an eigenvalue of operator A^* with algebraic multiplicity one.

Theorem 4.6 The system operator A generates a positive C_0 semigroup of contraction $T(t)$.

Theorem 4.7 The system $P(t,\cdot) = (P_0(t), P_1(t,x), P_2(t), P_{21}(t), P_{22}(t,y), P_3(t,x), P_{31}(t,x), P_{32}(t,x))^T$ has a unique nonnegative time-dependent solution $P(t,\cdot)$ with expression as

$$P(t,\cdot) = T(t)P_0, \quad \forall t \in [0,\infty).$$

Lemma 4.8 Assume that

$$0 < \hat{r} = \lim_{x\to\infty}\frac{1}{x}\int_0^x r(s)ds,$$
$$\hat{\mu} = \lim_{y\to\infty}\frac{1}{y}\int_0^y \mu(s)ds < \infty.$$

Then A_0 generates a quasi-compact semigroup $T_0(t)$.

For $\gamma > 0, P \in X$, let

$$(\Phi_\gamma(P))(x,y) = [diag(0, \varepsilon P_0 + \varepsilon P_{21}, 0, 0, \lambda_2 P_2 + \int_0^\infty r(s)P_{32}(s)ds,\ 0,\ 0,\ 0)] \cdot E_\gamma(x,y),$$

where

$$E_\gamma(x,y) = (0, e^{-\int_0^x(\gamma+\alpha_0+r(s))ds}, 0, e^{-\int_0^x(\gamma+\lambda_1+\lambda_2+r(s))ds}, e^{-\int_0^y(\gamma+\mu(s))ds}, e^{-\int_0^x(\gamma+r(s))ds}, e^{-\int_0^x(\gamma+r(s))ds})^T \in Ker(\gamma I - \overline{A}).$$

It is not hard to see that Φ_γ is a compact operator with the property that $I + \Phi_\gamma$ is a bijection from $D(A_0)$ to $D(A)$ and

$$[\gamma I - (A - B)]\ (I + \Phi_\gamma) = \gamma I - A_0.$$

Lemma 4.9 $S(t) - T_0(t)$ is a nonnegative compact operator, for any $t \geq 0$.

Theorem 4.10 C_0 semigroup $T(t)$ generated by the system operator A is quasi-compact.

5. Conclusion

In this paper, we discussed the existence and uniqueness of system solution and some properties of the system and also investigated the stability, especially the exponential stability of the system solution by C_0-semigroup theory.

References

[1] K. J. Engel and R. Nagel, (2000), One-Parameter Semigroups for Linear Evolution Equations, 194, Graduate Texts in Mathematics, Springer, Berlin, Germany.

[2] G. Geni, X. Z. Li, and G. T. Zhu, (2001), Functional Analysis Method in Queueing Theory, Research Information, West Berkshire, UK.

[3] W. H. Guo, (2002), The qualitative analysis of a system consisting of one reliable machine, one unreliable machine and a single storage buffer,

Mathematics in Practice and Theory, 32(6), 939 – 943.

[4] L. N. Guo, H. B. Xu, C. Gao and G. T. Zhu, (2011), Stability analysis of a new kind n −unit series repairable system, Appl. Math. Model., 35, 202 – 217.

[5] L. Guo, Z. Zhao, H. Xu, and G. Zhu, (2011), Reliability and proft analysis of a series repairable system with repairman vacation, Proceedings of the 30th Chinese Control Conference, 1009 – 1014.

[6] G. Gupur, X. Z. Li, and G. T. Zhu, (2001), An application of C_0 semigroup in queueing system, Semigroup Forum, 62(2), 205 – 216.

[7] J. C. Ke, C. H. Wu, and Z. G. Zhang, (2010), Recent developments in vacation models: a short survey, International Journal of Operations Research, 7(4), 3 – 8.

[8] Y. X. Li, (2005), Existence and uniqueness of positive periodic solutions for abstract semilinear evolution equations, J. Syst. Sci. Math. Sci., 28, 720 – 728.

[9] Y. X. Li, (2005), Existence of Solutions of Initial Value Problems for Abstract Semilinear Evolution Equations, Acta Math. Sinica, 48, 1089 – 1094.

[10] N. Tian and Z. G. Zhang, (2006), Vacation Queueing Models-Theory and Applications, 93, International Series in Operations Research & Management Science, Springer, New York, NY, USA.

[11] L. L. Wang, (2001), Studies of some aspects on abstract semilinear evolution solutions and functional differential equations, College of Mathematics and Econometrics, Hunan University.