On the Homogeneous Ternary Quadratic Diophantine Equation

$$3(x+y)^2-2xy=12z^2$$

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Abstract – *The ternary quadratic equation given by $3(x+y)^2-2xy=12z^2$ is considered and searched for its many different integer solutions. Eight different choices of integer solutions of the above equations are presented. A few interesting relations between the solutions and special polygonal numbers are presented.*

Key Words: **ternary quadratic, integer solutions. MSC subject classification: 11D09.**

1. INTRODUCTION

The diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-15] for quadratic equations with three unknowns.

This communication concerns with yet another interesting equation $3(x+y)^2-2xy=12z^2$ representing homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, few interesting relations among the solutions are presented.

2. Notations

- $t_{m,n}=n^{th}$ term of a regular polygon with m sides.
  $$=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$
- Triangular number of rank n, $T_{3,n}=\frac{n(n+1)}{2}$

3. Method of Analysis:

The ternary quadratic diophantine equation to be solved for its non-zero distinct integral solution is

$$3(x+y)^2-2xy=12z^2 \quad (1)$$

Note that (1) is satisfied by the following non-zero integer solutions.

$(-12ab, 2a^2-10b^2+8ab, a^2+5b^2)$,

$\begin{pmatrix}-18A^2+90B^2-72AB, \\ 24A^2-120B^2-12AB, 9A^2+45B^2\end{pmatrix}$,

$\begin{pmatrix}-112A^2+560B^2-308AB, \\ 126A^2-630B^2-168AB, 49A^2+245B^2\end{pmatrix}$,

$(2a^2-10b^2-8ab, 12ab, a^2+5b^2)$,

$\begin{pmatrix}-216A^2+1080B^2+108AB, \\ 126A^2-630B^2-792AB, 81A^2+405B^2\end{pmatrix}$,

$\begin{pmatrix}-1134A^2+5670B^2+1512AB, \\ 504A^2-2520B^2-4788AB, 441A^2+2205B^2\end{pmatrix}$

However, we have solutions for (1), which are illustrated below:

Introduction of the linear transformations $(u\neq v\neq 0)$

$$x=u+v,\ y=u-v \quad (2)$$

in (1) leads to

$$5u^2+v^2=6z^2 \quad (3)$$

Different patterns of solutions of (1) are presented below.

3.1. PATTERN-1

Write '6' as

$$6=(1+i\sqrt{5})(1-i\sqrt{5}) \quad (4)$$

Assume $z=a^2+5b^2$ (5)

where a and b are non- zero distinct integers.

Using (4) and (5) in (3), we get

$$5u^2+v^2=(1+i\sqrt{5})(1-i\sqrt{5})(a^2+5b^2)^2$$

Equating the positive and negative factors, the resulting equations are

$$v+i\sqrt{5}u=\left(1+i\sqrt{5}\right)\left(a+i\sqrt{5}b\right)^2 \quad (6)$$

$$v-i\sqrt{5}u=\left(1-i\sqrt{5}\right)\left(a-i\sqrt{5}b\right)^2 \quad (7)$$

Equating real and imaginary parts in (6), we get

$$u=a^2-5b^2+2ab$$

$$v=a^2-5b^2-10ab$$

Substituting the values of u and v in (2) we get,

$$x=x(a,b)=2a^2-10b^2-8ab \quad (8)$$

$$y=y(a,b)=12ab \quad (9)$$

Thus (8), (9) and (5) represent the distinct non-zero integral solutions of (1) in two parameters.

PROPERTIES:

- $x(a,1)+y(a,1)-4t_{3,a}-2\Pr_a+2t_{4,a}\equiv 0(\bmod 2)$
- $x(a,1)-y(a,1)-z(a,1)-4t_{3,a}+23\Pr_a-23t_{4,a}\equiv 0(\bmod 3)$
- $z(a,a+1)+4t_{4,a}-10\Pr_a\equiv 0(\bmod 5)$

3.2. PATTERN-2

Write '6' as

$$6=\frac{\left(7+i\sqrt{5}\right)\left(7-i\sqrt{5}\right)}{9} \quad (10)$$

Using (5) and (10) in (3), we get

$$5u^2+v^2=\frac{\left(7+i\sqrt{5}\right)\left(7-i\sqrt{5}\right)}{9}\left(a^2+5b^2\right)^2$$

Equating the positive and negative factors, the resulting equations are

$$v+i\sqrt{5}u=\frac{\left(7+i\sqrt{5}\right)}{3}\left(a+i\sqrt{5}b\right)^2 \quad (11)$$

$$v-i\sqrt{5}u=\frac{\left(7-i\sqrt{5}\right)}{3}\left(a-i\sqrt{5}b\right)^2 \quad (12)$$

Equating real and imaginary parts in (11), we get

$$u=\frac{1}{3}\left[a^2-5b^2+14ab\right]$$

$$v=\frac{1}{3}\left[7a^2-35b^2-10ab\right]$$

Replacing a and b by $3A$ and $3B$ respectively, we get

$$u=\frac{1}{3}\left[9A^2-45B^2+126AB\right]$$

$$v=\frac{1}{3}\left[63A^2-315B^2-90AB\right]$$

Substituting the values of u and v in (2) we get,

$$\left.\begin{aligned}x=x(A,B)&=24A^2-120B^2+12AB\\ y=y(A,B)&=-18A^2+90B^2+72AB\end{aligned}\right\} \quad (13)$$

and from (5) $z=z(A,B)=9\left(A^2+5B^2\right)$ (14)

Thus (13) and (14) represent the distinct non-zero integral solutions of (1) in two parameters.

PROPERTIES:

- $x(1,a)+z(1,a)+75\Pr_a-174t_{3,a}+87t_{4,a}\equiv 0(\bmod 3)$
- $x(a+1,1)+y(a+1,1)-12t_{3,a}-90\Pr_a+90t_{4,a}\equiv 0(\bmod 2)$
- $z(1,a)-y(1,a)+90t_{3,a}+27\Pr_a-27t_{4,a}=27$ is a cubical integer.

3.3. PATTERN-3

Write '6' as

$$6=\frac{\left(17+i\sqrt{5}\right)\left(17-i\sqrt{5}\right)}{49} \quad (15)$$

Using (5) and (15) in (3), we get

$$5u^2+v^2=\frac{\left(17+i\sqrt{5}\right)\left(17-i\sqrt{5}\right)}{49}\left(a^2+5b^2\right)^2$$

Equating the positive and negative factors, the resulting equations are

$$v+i\sqrt{5}u=\frac{\left(17+i\sqrt{5}\right)}{7}\left(a+i\sqrt{5}b\right)^2 \quad (16)$$

$$v-i\sqrt{5}u=\frac{\left(17-i\sqrt{5}\right)}{7}\left(a-i\sqrt{5}b\right)^2 \quad (17)$$

Equating real and imaginary parts in (16), we get

$$u=\frac{1}{7}\left[17a^2-85b^2-10ab\right]$$

$$v=\frac{1}{7}\left[a^2-5b^2+34ab\right]$$

Replacing a and b by $7A$ and $7B$ respectively, we get

$$u=\frac{1}{7}\left[833A^2-4165B^2-490AB\right]$$

$$v=\frac{1}{7}\left[49A^2-245B^2+1666AB\right]$$

Substituting the values of u and v in (2) we get,

$$\left.\begin{aligned} x=x(A,B)&=126A^2-630B^2+168AB\\ y=y(A,B)&=-112A^2+560B^2+308AB\end{aligned}\right\} \quad (18)$$

and from $z=z(A,B)=49\left(A^2+5B^2\right)$ (19)

Thus (18), and (19) represent the distinct non-zero integral solutions of (1) in two parameters.

PROPERTIES:

- $y(a,a+1)-z(a,a+1)-1918\Pr_a+917t_{4,a}\equiv 0(\bmod 5)$
- $x(1,a)+z(1,a)+770t_{3,a}-217\Pr_a+217t_{4,a}\equiv 0(\bmod 3)$
- $10\left(x(a,1)-y(a,1)-378t_{4,a}+140\Pr_a\right)\equiv 0(\bmod 119)$ is a cubical number.

3.4. PATTERN-4

One may write (3) as

$$5u^2+v^2=6z^2*1 \quad (20)$$

Write '1' as

$$1=\frac{\left(2+i\sqrt{5}\right)\left(2-i\sqrt{5}\right)}{9} \quad (21)$$

Using (4), (5) and (21) in (20), we get

$$5u^2+v^2=\frac{\left(17+i\sqrt{5}\right)\left(17-i\sqrt{5}\right)}{49}\frac{\left(2+i\sqrt{5}\right)\left(2-i\sqrt{5}\right)}{9}\left(a^2+5b^2\right)^2$$

Equating the positive and negative factors, the resulting equations are

$$v+i\sqrt{5}u=\frac{\left(17+i\sqrt{5}\right)}{7}\frac{\left(2+i\sqrt{5}\right)}{3}\left(a+i\sqrt{5}b\right)^2 \quad (22)$$

$$v-i\sqrt{5}u=\frac{\left(17-i\sqrt{5}\right)}{7}\frac{\left(2-i\sqrt{5}\right)}{3}\left(a-i\sqrt{5}b\right)^2 \quad (23)$$

Equating real and imaginary parts in (22), we get

$$u=\frac{1}{21}\left[19a^2-95b^2+58ab\right]$$

$$v=\frac{1}{21}\left[29a^2-145b^2-190ab\right]$$

Replacing a and b by $21A$ and $21B$ respectively, we get

$$u=\frac{1}{21}\left[8379A^2-41895B^2+22578AB\right]$$

$$v=\frac{1}{21}\left[12789A^2-63945B^2-83790AB\right]$$

Substituting the values of u and v in (2) we get,

$$\left.\begin{aligned} x=x(A,B)&=1008A^2-5040B^2-2772AB\\ y=y(A,B)&=-210A^2+1050B^2+520AB\end{aligned}\right\} \quad (24)$$

and from $z=z(A,B)=441\left(A^2+5B^2\right)$ (25)

Thus (24), and (25) represent the distinct non-zero integral solutions of (1) in two parameters.

PROPERTIES:

- $x(1,a)-y(1,a)-z(1,a)+315t_{4,a}+15960t_{3,a}\equiv 0(\bmod 7)$
- $x(a,a+1)-z(a,a+1)-8694t_{4,a}+17703\Pr_a\equiv 0(\bmod 3)$
- $y(a,1)+z(a,1)-462t_{3,a}-4977\Pr_a+4977t_{4,a}\equiv 0(\bmod 3)$

3.5. PATTERN-5

$$5u^2+v^2=6z^2$$

$$\left(z^2-v^2\right)=5\left(u^2-z^2\right)$$

$$(z+v)(z-v)=5(u+z)(u-z) \quad (26)$$

Equation (26) is written in the form of ratio as

$$\frac{z+v}{5(u-z)}=\frac{u+z}{z-v}=\frac{p}{q},q\neq 0 \quad (27)$$

From the First and third factors of (27), we have

$$\frac{z+v}{5(u-z)}=\frac{p}{q}$$

$$(z+v)q-5(u-v)p=0 \quad (28)$$

From the second and third factors of (28), we have

$$\frac{u+z}{(z-v)}=\frac{p}{q}$$

$$(u+z)q-(z-v)p=0 \quad (29)$$

Applying the method of cross multiplication for solving (28) and (29),

$$u=-5p^2+q^2-2pq$$

$$v=-5p^2+q^2+10pq$$

$$z=-5p^2-q^2$$

Substituting the values of u and v in (2) we get

$$\left.\begin{aligned} x=x(p,q)&=-10p^2+2q^2+8pq \\ y=y(p,q)&=-12pq \end{aligned}\right\} \quad (30)$$

Thus (30) along with the value of z represent the integer solutions to (1)

PROPERTIES:

- $x(1,a)-z(1,a)-5\Pr_a-6t_{3,a}+5t_{4,a}\equiv 0(\bmod 5)$
- $x(a,1+a)+y(a,1+a)+z(a,1+a)+20t_{4,a}\equiv 0(\bmod 1)$
- $y(a,1)+z(a,1)+24t_{3,a}-7t_{4,a}\equiv 0(\bmod 1)$

3.6. PATTERN-6

$$(v^2-u^2)=6(z^2-u^2)$$

$$(v+u)(v-u)=6(z+u)(z-u) \quad (31)$$

One may write equation (31) in the form of ratio as

$$\frac{v+u}{6(z-u)}=\frac{z+u}{v+u}=\frac{p}{q},q\neq 0 \quad (32)$$

From the First and third factors of (32), we have

$$\frac{v-u}{6(z-u)}=\frac{p}{q}$$

$$u(q+6p)+vq-6zp=0 \quad (33)$$

From the second and third factors of (32), we have

$$\frac{z+u}{(v-u)}=\frac{p}{q}$$

$$u(q+p)-vp+zq=0 \quad (34)$$

Applying the method of cross multiplication for solving (33) and (34),

$$u=-6p^2+q^2$$

$$v=-6p^2-q^2-12pq$$

$$z=-6p^2-q^2-2pq$$

Substituting the values of u and v in (2) we get

$$\left.\begin{aligned} x=x(p,q)&=-12p^2-12pq \\ y=y(p,q)&=2q^2+12pq \end{aligned}\right\} \quad (35)$$

Thus (35) along with the value of z represent the integer solutions to (1)

PROPERTIES:

- $x(a,1+a)-y(a,1+a)+28\Pr_a+10t_{4,a}\equiv 0(\bmod 2)$
- $z(1,a+1)+8t_{3,a}-3t_{4,a}\equiv 0(\bmod 3)$
- $y(1,a)-z(1,a)+11t_{4,a}-14\Pr_a=0$ is a nasty number.

3.7. PATTERN-7

Equation (3) is written in the form of ratio as

$$\frac{u+z}{2(u-z)}=\frac{3(u+z)}{u-v}=\frac{p}{q},q\neq 0 \quad (36)$$

From the First and third factors of (32), we have

$$\frac{u+v}{2(u-v)}=\frac{p}{q}$$

$$\Rightarrow u(q-2p)+vq+2zp=0 \qquad (37)$$

From the second and third factors of (36), we have

$$\frac{3(u+z)}{(u-v)}=\frac{p}{q}$$

$$\Rightarrow u(3q-p)+vp+3zq=0 \qquad (38)$$

Applying the method of cross multiplication for solving (37) and (38),

$$u=-2p^2+3q^2$$

$$v=-2p^2-3q^2+12pq$$

$$z=-2p^2-3q^2+2pq$$

Substituting the values of u and v in (2) we get

$$\left.\begin{array}{l} x=x(p,q)=-4p^2+12pq \\ y=y(p,q)=6q^2-12pq \end{array}\right\} \qquad (39)$$

Thus (39) along with the value of z represent the integer solutions to (1)

PROPERTIES:

- $z(a,1+a)+\Pr_a+6t_{3,a}-t_{4,a}\equiv 0(\bmod 3)$
- $x(1,a)-y(1,a)+z(1,a)-26\Pr_a+35t_{4,a}=0$ is a nasty number.
- $z(a,1)-x(a,1)+20t_{3,a}-12t_{4,a}\equiv 0(\bmod 3)$

3.8. PATTERN-8

Equation (3) can be written as

$$6z^2-v^2=5u^2 \qquad (40)$$

Assume $u=6a^2-b^2$ (41)

Write '5' as

$$5=(\sqrt{6}+1)(\sqrt{6}-1) \qquad (42)$$

Using (41), (42) in (40) and employing the method of factorization the above equation (40) is written as

$$(\sqrt{6}z+v)(\sqrt{6}z-v)=(\sqrt{6}a+b)^2(\sqrt{6}a-b)^2(\sqrt{6}+1)(\sqrt{6}-1)$$

Equating positive and negative factors, the resulting equations are

$$(\sqrt{6}z+v)=(\sqrt{6}a+b)^2(\sqrt{6}+1) \qquad (43)$$

$$(\sqrt{6}z-v)=(\sqrt{6}a-b)^2(\sqrt{6}-1) \qquad (44)$$

Equating rational and irrational parts in (43), we get

$$v=6a^2+b^2+12ab$$

$$z=6a^2+b^2+2ab \qquad (45)$$

Substituting the values of u and v in (2) we get

$$x=x(a,b)=12a^2+12ab \qquad (46)$$

$$y=y(a,b)=-2b^2-12ab \qquad (47)$$

Thus (45), (46) and (47) represent the distinct non-zero integral solutions of (1) in two parameters.

PROPERTIES:

- $x(1,a+1)-y(1,a+1)-28\Pr_a+26t_{4,a}\equiv 0(\bmod 2)$
- $y(1,a)+z(1,a)-9t_{4,a}-20t_{3,a}=0$ is a nasty number.
- $z(a,a+1)-18t_{3,a}+5\Pr_a-5t_{4,a}\equiv 0(\bmod 1)$

4. REMARKABLE OBSERVATIONS:

Let (u_0,v_0,z_0) be any given integer solution of (3), Then, each of the following triples of non-zero distinct integers based on u_0,v_0,z_0 also satisfies (1).

4.1. Triple 1: (u_0+h,v_0,z_0+h)

Here,

$$x_n=\frac{1}{2}\{(12-(-1)^n10)u_0+(-12+(-1)^n12)z_0\}+v_0$$

$$y_n=\frac{1}{2}\{(12-(-1)^n10)u_0+(-12+(-1)^n12)z_0\}-v_0$$

$$z_n=\frac{1}{2}\{(10-(-1)^n10)u_0+(-10+(-1)^n12)z_0\}$$

4.2. Triple 2: $\left(h-6u_0, h-6v_0, 6z_0\right)$

Here,

$$x_n = \frac{1}{36}\left\{\left(20(6)^n - 8(-6)^n\right)u_0 + \left(4(6)^n + 8(-6)^n\right)v_0\right\}$$

$$y_n = \frac{1}{2}\left\{\left(12(-1)^n\right)u_0 - \left(12(-6)^n\right)v_0\right\}$$

$$z_n = 6^n z_0$$

4.3. Triple3: $\left(8u_0, -8v_0 + h, 8z_0 + h\right)$

Here,

$$x_n = 8^n u_0 + \frac{1}{32\sqrt{6}}\left\{16\sqrt{6}A_n v_0 + 96B_n z_0\right\}$$

$$y_n = 8^n u_0 - \frac{1}{32\sqrt{6}}\left\{16\sqrt{6}A_n v_0 + 96B_n z_0\right\}$$

$$z_n = \frac{1}{32\sqrt{6}}\left\{16B_n v_0 + 16\sqrt{6}A_n z_0\right\}$$

where $A_n = \left(40+16\sqrt{6}\right)^n + \left(40-16\sqrt{6}\right)^n$

$$B_n = \left(40+16\sqrt{6}\right)^n - \left(40-16\sqrt{6}\right)^n$$

4.4. Triple 4: $\left(-3u_0 + h, 3v_0, 3z_0 + h\right)$

Here,

$$x_n = \frac{1}{12\sqrt{30}}\left\{6\sqrt{30}A_n u_0 - 36B_n z_0\right\} + 3^n v_0$$

$$y_n = \frac{1}{12\sqrt{30}}\left\{6\sqrt{30}A_n u_0 - 36B_n z_0\right\} - 3^n v_0$$

$$z_n = \frac{1}{12\sqrt{30}}\left\{-30B_n u_0 + 6\sqrt{30}A_n z_0\right\}$$

Where $A_n = \left(-33+6\sqrt{30}\right)^n + \left(-33-6\sqrt{30}\right)^n$

$$B_n = \left(-33+6\sqrt{30}\right)^n - \left(-33-6\sqrt{30}\right)^n$$

5. CONCLUSION

In this paper, we have presented infinitely many non-zero distinct integer solutions to the ternary quadratic equation

$$3(x+y)^2 - 2xy = 12z^2$$

representing a homogeneous cone. As diophantine equation are rich in variety, to conclude, one may search for other forms of three dimensional surfaces, namely, non-homogeneous cone, paraboloid, ellipsoid, hyperbolic paraboloid and so on for finding integral points on them and corresponding properties.

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