Observations on the Non-homogeneous binary Quadratic Equation $8x^2 - 3y^2 = 20$

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Abstract – *A Non-homogeneous binary quadratic equation represents hyperbola given by $8x^2 - 3y^2 = 20$ is analyzed for its non-zero distinct integer solutions. A few interesting relation between the solution of the given hyperbola, integer solutions for other choices of hyperbola and parabola are obtained.*

***Key Words*: Non-homogeneous quadratic, binary quadratic, integer solutions.**

1. INTRODUCTION

The binary quadratic Diophantine equations of the form $ax^2 - by^2 = N, (a, b, N \neq 0)$ are rich in variety and have been analysed by many mathematicians for their respective integer solutions for particular values of a , b and N. In this context, one may refer [1-13].

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $8x^2 - 3y^2 = 20$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented.

2. Method of Analysis

The Diophantine equations representing the binary quadratic equation to be solved for its non-zero distinct integer solution is

$$8x^2 - 3y^2 = 20 \quad (1)$$

Consider the linear transformations

$$x = X + 3T \text{ , } y = X + 6T \quad (2)$$

From(1) and(2), we have

$$X^2 = 30T^2 + 19 \quad (3)$$

Whose smallest positive integer solution is

$$X_0 = 7, T_0 = 1$$

To obtain the other solutions of (3), consider the Pell equation

$$X^2 = 30T^2 + 1 \quad (4)$$

Whose smallest positive integer solution is $\left(\tilde{X}_0, \tilde{T}_0\right) = (1,5)$

The general solution of (4) is given by

$$\tilde{T}_n = \frac{1}{2\sqrt{30}} g_n , \tilde{X}_n = \frac{1}{2} f_n$$

where

$$f_n = (5+\sqrt{24})^{n+1} + (5-\sqrt{24})^{n+1}$$

$$g_n = (5+\sqrt{24})^{n+1} - (5-\sqrt{24})^{n+1}, n = -1,0,1,\ldots.,$$

Applying Brahmaguptha lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solutions of (1) are given by,

$$x_{n+1} = 8 f_n + \frac{39}{\sqrt{24}} g_n$$

$$y_{n+1} = 13 f_n + \frac{64}{\sqrt{24}} g_n$$

The recurrence relations satisfied by and y are given by

$$x_{n+3} - 10x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 10y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table :1 below

Table:1 Numerical example

n	x_n	y_n
0	16	26
1	158	258
2	1564	2554
3	15482	25282
4	153256	250266

From the above table, we observe some interesting relations among the solutions which are presented below:

- Both x_n and y_n values are even .

2.1. Relations among the solutions are given below.

- $3y_{n+1} - x_{n+2} + 5x_{n+1} = 0$
- $x_{n+3} - 10x_{n+2} + x_{n+1} = 0$
- $3y_{n+2} - 5x_{n+2} + x_{n+1} = 0$

- $3y_{n+3} - 49x_{n+2} + 5x_{n+1} = 0$
- $30y_{n+1} - x_{n+3} + 49x_{n+1} = 0$
- $49y_{n+1} - y_{n+3} + 80x_{n+1} = 0$
- $49y_{n+2} - 5y_{n+3} + 8x_{n+1} = 0$
- $5x_{n+3} - 49x_{n+2} - 3y_{n+1} = 0$
- $5y_{n+2} - 8x_{n+2} - y_{n+1} = 0$
- $y_{n+3} - 16x_{n+2} - y_{n+1} = 0$
- $49y_{n+2} - 8x_{n+3} - 5y_{n+1} = 0$
- $49y_{n+3} - 80x_{n+3} - y_{n+1} = 0$
- $8x_{n+1} - y_{n+2} + 5y_{n+1} = 0$
- $y_{n+3} - 10y_{n+2} + y_{n+1} = 0$
- $3y_{n+3} - 5x_{n+3} + x_{n+2} = 0$
- $x_{n+3} - 5x_{n+2} - 3y_{n+2} = 0$
- $y_{n+3} - 8x_{n+2} - 5y_{n+2} = 0$
- $5y_{n+3} - 8x_{n+3} - y_{n+2} = 0$
- $6y_{n+3} - x_{n+3} + x_{n+1} = 0$
- $30y_{n+3} - 49x_{n+3} + x_{n+1} = 0$

2.2. Each of the following expression is a nasty number:

- $\frac{1}{5}[60 + 384x_{2n+2} - 234y_{2n+2}]$
- $\frac{1}{15}[180 + 2322x_{2n+2} - 234x_{2n+3}]$
- $\frac{1}{150}[1800 + 22986x_{2n+2} - 234x_{2n+4}]$
- $\frac{1}{25}[300 + 3792x_{2n+2} - 234y_{2n+3}]$
- $\frac{1}{245}[2940 + 37536x_{2n+2} - 234y_{2n+4}]$
- $\frac{1}{25}[600 + 384x_{2n+3} - 2322y_{2n+2}]$
- $\frac{1}{245}[2940 + 384x_{2n+4} - 22986y_{2n+2}]$
- $\frac{1}{40}[480 + 384y_{2n+3} - 3792y_{2n+2}]$
- $\frac{1}{400}[4800 + 384y_{2n+4} - 37536y_{2n+2}]$
- $\frac{1}{15}[180 + 22986x_{2n+3} - 2322x_{2n+4}]$
- $\frac{1}{5}[60 + 3792x_{2n+3} - 2322y_{2n+3}]$
- $\frac{1}{25}[300 + 37536x_{2n+3} - 2322x_{2n+4}]$
- $\frac{1}{25}[300 + 3792x_{2n+4} - 22986y_{2n+3}]$

- $\frac{1}{5}\left[60+37536x_{2n+4}-22986x_{2n+3}\right]$
- $\frac{1}{40}\left[480+3792y_{2n+4}-37536y_{2n+3}\right]$

2.3. Each of the following expressions is a cubical integer.

- $\frac{1}{5}\left[64x_{3n+3}-39y_{3n+3}+192x_{n+1}-117y_{n+1}\right]$
- $\frac{1}{15}\left[387x_{3n+3}-39x_{3n+4}+1161x_{n+1}-117x_{n+2}\right]$
- $\frac{1}{150}\left[3831x_{3n+3}-39x_{3n+5}+11493x_{n+1}-117x_{n+3}\right]$
- $\frac{1}{25}\left[632x_{3n+3}-39y_{3n+4}+1896x_{n+1}-117y_{n+2}\right]$
- $\frac{1}{245}\left[6256x_{3n+3}-39y_{3n+5}+18768x_{n+1}-117y_{n+3}\right]$
- $\frac{1}{25}\left[64x_{3n+4}-387y_{3n+3}+192x_{n+2}-1161y_{n+1}\right]$
- $\frac{1}{245}\left[64x_{3n+5}-3831y_{3n+3}+192x_{n+3}-11493y_{n+1}\right]$
- $\frac{1}{40}\left[64y_{3n+4}-632y_{3n+3}+192y_{n+2}-1896y_{n+1}\right]$
- $\frac{1}{400}\left[64y_{3n+5}-6256y_{3n+3}+192y_{n+3}-18768y_{n+1}\right]$
- $\frac{1}{15}\left[3831x_{3n+4}-387x_{3n+5}+11493x_{n+2}-1161x_{n+3}\right]$
- $\frac{1}{5}\left[632x_{3n+4}-387y_{3n+4}+1896x_{n+2}-1161y_{n+2}\right]$
- $\frac{1}{25}\left[6256x_{3n+4}-387y_{3n+5}+18768x_{n+2}-1161y_{n+2}\right]$
- $\frac{1}{25}\left[632x_{3n+5\backslash}-3831y_{3n+4}+1896x_{n+3}-11493y_{n+2}\right]$
- $\frac{1}{5}\left[6256x_{3n+5}-3831y_{3n+5}+18768x_{n+3}-11493y_{n+3}\right]$
- $\frac{1}{40}\left[632y_{3n+5}-6256y_{3n+4}+1896y_{n+3}-18768y_{n+2}\right]$

2.4. Each of the following expressions is a biquadratic integer.

- $\frac{1}{5}\left[64x_{4n+4}-39y_{4n+4}+256x_{2n+2}-156y_{2n+2}+30\right]$
- $\frac{1}{15}\left[387x_{2n+2}-39x_{4n+5}+1548x_{2n+2}-156x_{2n+3}+90\right]$
- $\frac{1}{150}\left[3831x_{4n+4}-39x_{4n+6}+15324x_{2n+2}-156x_{2n+4}+900\right]$
- $\frac{1}{25}\left[632x_{4n+4}-39y_{4n+5}+2528x_{2n+2}-156y_{2n+3}+150\right]$
- $\frac{1}{245}\left[6256x_{4n+4}-39y_{4n+6}+25024x_{2n+2}-156y_{2n+4}+1470\right]$

- $\frac{1}{25}[64x_{4n+5}-387y_{4n+4}+256x_{2n+3}-1548y_{2n+2}+150]$
- $\frac{1}{245}[64x_{4n+6}-3831y_{4n+4}+256x_{2n+4}-15324y_{2n+2}+1470]$
- $\frac{1}{40}[64y_{4n+5}-632y_{4n+4}+256y_{2n+3}-2528y_{2n+2}+240]$
- $\frac{1}{400}[64y_{4n+6}-6256y_{4n+4}+256y_{2n+4}-25024y_{2n+2}+2400]$
- $\frac{1}{5}[3831x_{4n+5}-387x_{4n+6}+15324x_{2n+3}-1548x_{2n+4}+90]$
- $\frac{1}{5}[632x_{4n+5}-387y_{4n+5}+2528x_{2n+3}-1548y_{2n+3}+30]$
- $\frac{1}{25}[6256x_{4n+5}-387y_{4n+6}+25024x_{2n+3}-1548y_{2n+4}+150]$
- $\frac{1}{25}[632x_{4n+6}-3831y_{4n+6}+25624x_{2n+4}-15324y_{2n+4}+30]$
- $\frac{1}{5}[6256x_{4n+6}-3831y_{4n+6}+25624x_{2n+4}-15324y_{2n+4}+30]$
- $\frac{1}{40}[632y_{4n+6}-6256y_{4n+5}+2528y_{2n+4}-25024y_{2n+3}+240]$

2.5.Each of the following expression is a quintic integer.

- $\frac{1}{5}[64x_{5n+5}-39y_{5n+5}+320x_{3n+3}-195y_{3n+3}+640x_{n+1}-390y_{n+1}]$
- $\frac{1}{15}[387x_{5n+5}-39x_{5n+6}+1935x_{3n+3}-195x_{3n+4}+3870x_{n+1}-390x_{n+2}]$
- $\frac{1}{150}[3831x_{5n+5}-39x_{5n+7}+19155x_{3n+3}-195x_{3n+5}+38310x_{n+1}-390x_{n+3}]$
- $\frac{1}{25}[632x_{5n+5}-39y_{5n+6}+3160x_{3n+3}-195y_{3n+4}+6320x_{n+1}-390y_{n+2}]$
- $\frac{1}{245}[6256x_{5n+5}-39y_{5n+7}+31280x_{3n+3}-195y_{3n+5}-62560x_{n+5}-390x_{n+3}]$
- $\frac{1}{25}[64x_{5n+6}-387y_{5n+5}+320x_{3n+4}-1935y_{3n+3}+640x_{n+2}-3870y_{n+1}]$
- $\frac{1}{40}[64x_{5n+7}-632y_{5n+5}+320y_{3n+4}-3160y_{3n+3}+640y_{n+2}-6320y_{n+3}]$
- $\frac{1}{400}[64y_{5n+7}-6256y_{5n+5}+320y_{3n+5}+31280y_{3n+3}-640y_{n+3}-62560y_{n+1}]$
- $\frac{1}{15}[3831x_{5n+6}-387x_{5n+7}+19155x_{3n+4}-1932y_{3n+4}+38310x_{n+2}-3870x_{n+3}]$
- $\frac{1}{5}[632x_{5n+6}-387y_{5n+6}+3160x_{3n+4}-1932y_{3n+4}+6320x_{n+2}-3870y_{n+2}]$
- $\frac{1}{25}[6256x_{5n+6}-387y_{5n+7}+31280x_{3n+4}-1935y_{3n+5}+62560x_{n+2}-3870y_{n+3}]$
- $\frac{1}{25}[632x_{5n+7}-3831y_{5n+6}+3160x_{3n+5}-19155y_{3n+4}+6320x_{n+3}-38310y_{n+2}]$

- $\frac{1}{5}[6256x_{5n+7}-3831y_{5n+7}+31280x_{3n+5}-19155y_{3n+5}+62560x_{n+2}-38310y_{n+3}]$
- $\frac{1}{40}[632y_{5n+7}-6256y_{5n+6}+3160y_{3n+5}-31280y_{3n+4}+6320y_{n+3}-62650y_{n+2}]$
- $\frac{1}{245}[64x_{5n+7}-3831y_{5n+5}+320x_{3n+5}-19155y_{3n+3}+640x_{n+3}-38310y_{n+1}]$

REMARKABLE OBSERVATIONS

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table:2 below:

Table:2 Hyperbolas

S.NO	Hyperbola	(X,Y)
1	$Y^2-24X^2=100$	$(8y_{n+1}-13x_{n+1},64x_{n+1}-39y_{n+1})$
2	$Y^2-24X^2=900$	$(8x_{n+2}-79x_{n+1},387x_{n+1}-39x_{n+2})$
3	$Y^2-24X^2=90000$	$(8x_{n+3}-782x_{n+1},3831x_{n+1}-39x_{n+3})$
4	$Y^2-24X^2=2500$	$(8y_{n+2}-129x_{n+1},632x_{n+1}-39y_{n+2})$
5	$Y^2-24X^2=240100$	$(8y_{n+3}-1277x_{n+1},6256x_{n+1}-39y_{n+3})$
6	$Y^2-24X^2=2500$	$(79y_{n+1}-13x_{n+2},64x_{n+2}-387y_{n+3})$
7	$Y^2-24X^2=240100$	$(782y_{n+1}-13x_{n+3},64y_{n+2}-3831y_{n+1})$
8	$Y^2-24X^2=6400$	$(129y_{n+1}-13y_{n+2},64y_{n+2}-632y_{n+1})$
9	$Y^2-24X^2=640000$	$(1277y_{n+1}-13y_{n+3},64y_{n+3}-6256y_{n+1})$
10	$Y^2-24X^2=900$	$(79x_{n+3}-782x_{n+2},3831x_{n+2}-387x_{n+3})$
11	$Y^2-24X^2=100$	$(79y_{n+2}-129x_{n+2},632x_{n+2}-387y_{n+2})$
12	$Y^2-24X^2=2500$	$(79y_{n+3}-1277x_{n+2},6256x_{n+2}-387y_{n+3})$
13	$Y^2-24X^2=2500$	$(782y_{n+2}-129x_{n+3},632x_{n+3}-3831y_{n+2})$
14	$Y^2-24X^2=100$	$(782y_{n+3}-1277x_{n+3},6256x_{n+3}-3831y_{n+3})$
15	$Y^2-24X^2=6400$	$(1277y_{n+2}-129y_{n+3},632y_{n+3}-6256y_{n+2})$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table: 3 below:

Table: 3 Parabolas

S.N0	Parabola	(X,Y)
1	$5Y-24X^2=100$	$(8y_{n+1}-13x_{n+1},64x_{2n+2}-39y_{2n+2}+10)$
2	$15Y-24X^2=900$	$(8x_{n+2}-79x_{n+1},387x_{2n+2}-39x_{2n+3}+30)$
3	$150Y-24X^2=90000$	$(8x_{n+3}-782x_{n+1},3831x_{2n+2}-39x_{2n+4}+300)$
4	$25Y-24X^2=2500$	$(8y_{n+2}-129x_{n+1},632x_{2n+2}-39y_{2n+3}+50)$
5	$245Y-24X^2=240100$	$(8y_{n+3}-1277x_{n+1},6256x_{2n+2}-39y_{2n+4}+490)$
6	$25Y-24X^2=2500$	$(79y_{n+1}-13x_{n+2},64x_{2n+3}-387y_{2n+2}+50)$
7	$245Y-24X^2=240100$	$(782y_{n+1}-13x_{n+3},64x_{2n+4}-3831y_{2n+2}+490)$
8	$40Y-24X^2=6400$	$(129y_{n+1}-13y_{n+2},64y_{2n+3}-632y_{2n+2}+80)$
9	$400Y-24X^2=640000$	$(1277y_{n+1}-13y_{n+3},64y_{2n+4}-6256y_{2n+2}+800)$
10	$15Y-24X^2=90$	$(79x_{n+3}-782x_{n+2},3831x_{2n+3}-387x_{2n+4}+30)$
11	$5Y-24X^2=100$	$(79y_{n+2}-129x_{n+2},632x_{2n+3}-387y_{2n+3}+10)$
12	$25Y-24X^2=2500$	$(79y_{n+2}-1277x_{n+},6256x_{2n+3}-387y_{2n+4}50)$
13	$25Y-24X^2=2500$	$(782y_{n+2}-129x_{n+3},632x_{2n+4}-3831y_{2n+3}+50)$
14	$5Y-24X^2=100$	$(782y_{n+3}-1277x_{n+3},6256x_{2n+4}-3831y_{2n+4}+10)$
15	$40Y-24X^2=6400$	$(1277y_{n+2}-129y_{n+3},632y_{2n+4}-6256y_{2n+4}+1610)$

3. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the Diophantine equation, represented by hyperbola is given by $8x^2-3y^2=20$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties.

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