Ordered Meet Hyperlattices

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ABSTRACT - *In this paper we introduce the notion of the notion of meet hyperlattice in the Ordered hyperlattices. Also, we discuss about the product of two ordered meet hyperlattices. We investigate the properties of prime ideals in the product of two ordered meet hyperlattices.*

Keywords: *hyperlattices, ordered hyperlattice, meet hyperlattice and prime ideals*

INTRODUCTION:

Hyperlattices are the most advancing and developing area in lattice theory. Ordered hyperlattices has been introduced. The notion of meet hyperlattice is introduced to the ordered hyperlattices in this paper. We also define the prime ideals in the product of two ordered meet hyperlattices and we investigate about their properties.

I. Some basic definitions and Properties:

Definition 1.1:

Let H be a non-empty set. A Hyperoperation on H is a map ∘ from H×H to P*(H), the family of non-empty subsets of H. The Couple (H, ∘) is called a hypergroupoid. For any two non-empty subsets A and B of H and x Є H, we define A ∘ B = $\cup_{a\in A, b\in B}$ a ∘ b;

A ∘ x = A ∘ {x} and x ∘ B = {x} ∘ B

A Hypergroupoid (H, ∘) is called a Semihypergroup if for all a, b, c of H we have (a ∘ b) ∘ c = a ∘ (b ∘ c). Moreover, if for any element a ∈ H equalities

A ∘ H = H ∘ a = H holds, then (H, ∘) is called a Hypergroup.

Definition 1.2:

A Lattice [1] is a partially ordered set L such that for any two elements x,y of L, glb {x, y} and lub {x, y} exists. If L is a lattice, then we define x V y = glb {x, y} and lub {x,y}.

Proposition 1.3:

The Definition of Lattice is equal to the following propositions. Let L be a non-empty set with two binary operations ∧ and V. Let for all a, b, c Є L, the following conditions holds

1) a ∧ a = a and a V a = a
2) a ∧ b = b ∧ a and a V b = b V a
3) (a ∧ b) ∧ c = a ∧ (b ∧ c) and (a V b) V c = a V (b V c)
4) (a ∧ b) V a = a and (a V b) ∧ a = a

Then (L, V, ∧) is a Lattice.

Definition 1.4:

Let L be a non-empty set, ? : L × L → p* (L) be a hyperoperation and V : L × L →L be an operation. Then (L, V, ?) is a Meet Hyperlattice [3], if for all x, y, z Є L. The following conditions are satisfied:

1) x Є x ? x and x = x V x
2) x V (y V z) = (x V y) V z and
3) x ? (y ? z)=(x ? y) ? z
4) x V y = y V x and x ? y = y ? x
5) x Є x ? (x V y) ∩ x V (x? y)

Definition 1.5:

Let L be both join and meet hyperlattice (which means that V and ∧ are both hyperoperations), then we call L as Total Hyperlattice.[2]

Example 1.6:

Let (L, ≤) be a partial ordered set. We define the operation V as,

a V b = { x Є L : x ≤ a, x ≤ b}

and the hyperoperation ? as,

a ? b = {x Є L : a ≤ x, b ≤ x}.

Then, (L, V, ?) is a Meet Hyperlattice.

Definition 1.7:

Let I be a non-empty subset of L. Then I is called an Ideal [4] of L if

1) for every x, y Є I, x V y C I
2) x ≤ I implies x Є I

Now, let (L, V, ?) be a meet hyperlattice. We call (L, ≤) is an ordered hyperlattice, if ≤ is an equivalence relation and x ≤ y implies that x ? z ≤ y ? z and x V z ≤ y V z.

Note that for any A, B C L, A ≤ B means that there exist x Є A, y Є B such that x ≤ y.

Definition 1.8:

Let L be a non-empty set, ? be a binary operation and V be a hyperoperation on L. L is called a Hyperlattice if for all a, b, c Є L, the following conditions holds:

1) a Є a V a and a ∧ a = a
2) a V b = b V a and a ∧ b = b ∧ a
3) a Є [a ∧ (a V b)] ∩ [a V (a ∧ b)]
4) a V (b V c) = (a V b) V c and a ∧ (b ∧ c) = (a ∧ b) ∧ c
5) a Є a V b implies a ∧ b = b

Let A, B C L. Then define

A V B = ∪ {a V b | a Є A, b Є B}

A ∧ B = {a ∧ b | a Є A, b Є B}

Note:

I L is a s-distributive hyperlattice, then L is an Ordered Hyperlattice but the converse is not true.

Example 1.9:

Let (L, V, ?) be a strong meet hyperlattice such that

x ? $y = x$? $x \cap y$? y *and*

if x ? $x = y$? y, *then* $x = y$.

We define the relation $\leq$ *as* $x \leq y$ *implies* $x \in y$? y. *Thus*, $(L, V,$? $, \leq)$ *is an ordered hyperlattice*.

***Definition* 1.10**:

Let $(L_1, V_1, \Lambda_1, \leq_1)$ and $(L_2, V_2, \Lambda_2, \leq_2)$ be two ordered hyperlattiice.

Give $(L_1 \times L_2, V', \Lambda')$, be two hyperoperations V' and ?' *on* $L_1 \times L_2$ *such that for any*

$(x_1, y_1), (x_2, y_2) \in L_1 \times L_2$, *we have*

(x_1, y_1) ?' $(x_2, y_2) = \{(u, v);\ u \in x_1 \Lambda_1 x_2, v \in y_1 \Lambda_2 y_2\}$,

$(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq_1 x_2$, $y_1 \leq_2 y_2$.

The Hyperoperation **V'** *is defined similar to* ?'.

Definition 1.11:

Let $\mathcal{R}$ be an equivalence relation on a non-empty set L and A, B C L, A $\bar{\mathcal{R}}$ B means that

for all a Є A, there exists some b Є B such that a $\mathcal{R}$ b,

for all b' Є B, there exists a' Є A such that a' $\mathcal{R}$ b'.

Also, $\mathcal{R}$ is calleda regular relation respect to ? if x $\mathcal{R}$ y implies that x ∧ z $\bar{\mathcal{R}}$ y ∧ z, or all x, y, z Є L. $\mathcal{R}$ is called a Regular relation if it is regular respect to V and ∧, at the same time.

Theorem 1.12:

Let (L, V, ?) be a hyperlattice and γ be an equivalence relation on L. Then, (L/γ, V, ∧) is a hyperlattice if and only if γ is a regular relation.

Proof:

Let us assume that (L/γ, V, ∧) is a Hyperlattice.

For some x, y Є L/γ then,

x γ y

since L/γ is distributive, for some z Є L/γ,

(x γ y) ∧ z implies (x ∧ z) $\bar{\gamma}$ (y ∧ z)

Similarly, (x γ y) V z implies (x V z) $\bar{\gamma}$ (y V z)

Therefore γ is a regular relation.

Conversely, assume that γ is a regular relation then for all x, y Є L/γ there exists x γ y.

Let A, B C L/γ.

x γ y implies (x V z) $\bar{\gamma}$ (y V z) that is (x γ y) V z for all z Є L/γ

Then, A V B = ∪ {(x γ y) V z | x γ y Є A, z Є B}

Similarly, the condition for the hyperoperation ∧ holds.

Therefore (L/γ, V, ∧) is a Hyperlattice.

II. **Properties of Prime Ideals in product of two ordered meet hyperlattices**

In this section, we consider strong meet hyperlattices. First we define prime ideals in strong meet hyperlattices. Then, we investigate sufficient conditions of a subset of product of two ordered strong meet hyperlattices is a prime ideal. Also, we define special elements

in ordered strong meet hyperlattices and we investigate the connection between this elements and ideals in ordered strong meet hyperlattice.

Definition 2.1:

An Ideal P of a meet hyperlattice L is Prime [5], if for all x, y Є L and x V y Є P, we have x Є P and y Є P.

Proposition 2.2:

Let L be a meet hyperlattice. A subset P of a hyperlattice L is prime if an only if L\P is a subhyperlattice of L.

Proof:

Let x, y Є L\P

Then, by the definition of prime ideal we have,

x V y does not belong to P

Therefore x V y Є L\P

We prove that x ? y C L\P

Let x ? y C P and x does not belong to P, Y does not belongs to P.

We have (x ? y) V x Є P

and

(x ? y) V y Є P [since P is an ideal of hyperlattice L]

Since L is a meet hyperlattice

we have x Є (x ? y) V x which implies that x Є P and y Є P

which is a contradiction.

Thus, x ? y is not the subset of P and x ? y C L\P

Similarly, we show that x V y Є L\P and L\P is a subhyperlattice of L.

Let L\P is a subhyperlattice of L and

x V y Є P, x and y does not belongs to P.

Tus x, y Є L\P, we have x V y Є L\P

Therefore x V y doesnot belongs to P and this is a contradiction.

Hence x Є P and y Є P.

Theorem 2.3:

Let (L_1, V_1, $\bigwedge_1$, $\leq_1$) and (L_2, V_2, $\bigwedge_2$, $\leq_2$) be two ordered strong meet hyperlattices and L C L_1×L_2. L is a prime ideal if and only if there exists a prime ideal I C L_1 and J C L_2 wih the properties that for any x Є L_1, x' Є L_2 and y Є I, y' Є J, we have x $\wedge_1$ y C I, x' $\wedge_2$ y' C J and L = (I × L_2) ∩ (L_1 × J).

Proof:

Let us first prove the converse part.

We show that L is a prime ideal of L_1× L_2.

Let (x, z), (y, w) Є L

If x, y Є I, we have x $\wedge_1$ y C I.

Since L_2 is a meet hyperlattice, we have z $\wedge_2$ w C L_2.

Thus, (x, z) ?' (y, w) = x $\wedge_1$ y × z $\wedge_2$ w C I × L_2 C L.

If z Є J, w Є L_2, we have,

(x, z) ?' (y, w) = x $\wedge_1$ y × z $\wedge_2$ w

C I × J C I × L_2 C L.

Let, (x, y) Є L_1×L_2, (z, w) Є L.

If z Є I and w Є L_2, we have

(x, z) V' (z, w) Є I × L_2 C L

and if z Є L_1 and w Є J, we have

(x, y) V' (z, w) Є L_1 × J C L.

Now, we show that L is prime.

Let (x, y) V' (z, w) Є L.

Then we have x V_1 z × y V_2 w Є L.

Therefore,

1) x V_1 z Є I, y V_2 w Є L_2 and
2) x V_2 z Є L_1, y V_2 w Є J.

since I is prime, in the first case, we have x Є I and z Є I.

Thus, (x, y) Є I × L_2 and (z, w) Є I × L_2.

In the second case the proof is similar.

Therefore L is a prime ideal.

Let L be a prime ideal and (x, y) Є L.

We show that {x} × L_2 C L and

L_1 × {z} C L.

If these two relations are not true, there exist y Є L_2 and w Є L_1 such that (x, y) does not belongs to L and (w, z) does not belongs to L. Since L is prime, we have (x V_1 w) × (y V_2 z) Є L which is a contradiction.

Now, we define

A = {x Є L_1; {x} × L_2 C L} and

B = {z Є L_2; L_1 × {z} C L}

and

I = {y Є L_1; y $\leq_1$ a for some a Є A} and

J = {y Є L_2; y $\leq_1$ b for some b Є B}

By the definitions of A, B, I and J, we can easily show that L = (I × L_1) ∩ (L_2 × J)

Now, we show that I is a prime ideal and it has the property which is stated in the assumption of theorem.

Let x, y Є L_1 and x V_1 y Є I.

Then there exists a Є A such that

x V_1 y $\leq_1$ a.

Also, (x, a) V'(y, a) = (x V_1 y, a V_2 a) Є I × L_1 C L.

Since L is a prime ideal, we have

(x, a) Є L and (y, a) Є L.

Therefore, since a Є L_1, we have x Є I and y Є I.

Let x Є L_1, y Є I, there exists some a Є A such that y $\leq_1$ a.

Thus x $\wedge_1$ y $\leq_1$ a $\wedge_1$ y.

Since a $\wedge_1$ y C I and I is an Ideal,

We have x $\wedge_1$ y Є I.

Hence the proof.

Definition 2.4:

Let (L, V, ∧, ≤) be an ordered meet hyperlattice which is not bounded and x Є L.

If (x V L] = $\cup_{y \in L}$ x V y = {y Є L; y ≤ a for some a Є x V L} = L, then we call x is right simple element. Now, let L is bounded with greatest element 1, x Є L is right simple element,

if (x V L] = L\{1}.

Theorem 2.5:

Let L be a distributive ordered strong meet hyperlattice and R be the set of all right simple elements of L. Then, we have R is a subhyperlattice of L. Also, if for any arbitrary subset A, B C L, y Є L we have y Є A ? B implies that y Є A and y Є B and L\R is non-empty, then L\R is a maximal ideal of L.

Proof:

Let a, b Є R.

Then we have (a V L] = L,

(b V L] = L and

L = (a V L] = (a V (b V L]] C (a V b V L].

Therefore, a V b Є R.

Since, (A] ? (B] C (A ? B] and L is distributive. We have,

L = (a V L] ? (b V L] C ((a ? b) V L].

So, a ? b Є R.

Let x, y ЄL\R.

If x ? y C R, we have L = ((x ? y) V L].

Let z Є L, then there exists z' Є (x ? y) V L such that z ≤ z'.

Therefore, there exists w Є L such that z' = (x ? y) V w which implies that

z' Є (x V w) ? (y V w).

By our assumption, we conclude that

z' Є x V w and z' Є y V w.

Therefore, z ≤ z' Є x V w and

z ≤ z' Є y ∧ w.

Then, z Є (x V L] and z Є (y V L] and

L = (x V L] and L = (y V L].

So x ? y is not the subset of R and x ? y C L\R.

Let x Є L\R and y Є L.

We show that x V y Є R, we have

L= (x V L] C (x V y V L] = L which is a contradiction.

Therefore, x V y Є L\R.

L\R is a maximal ideal of L.

III. Conclusion:

Hence we introduced the ordered meet hyperlattice and we depicted them in product. We also investigated the properties of prime ideals in the product of two ordered meet hyperlattices.

IV. References:

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