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On Greatest Common Divisor and its Application for a Geometrical Structure

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Abstract - *In this article some results about GCD are discussed and we derive a formula for smallest number of identical cuboids to construct a square floor/ a cube.*

***Key Words*: GCD, LCM, prime, relatively prime, cuboid.**

1. INTRODUCTION

$\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ are sets of natural numbers and integers respectively. A nonzero integer b is a divisor (factor) of $a \in \mathbb{Z}$ if $a = kb$ for some $k \in \mathbb{Z}$ and in this case we write $b|a$ and in this case $|b| \leq |a|$ if $a \neq 0$. Divides '|' is areflexive, transitive relation on a set of nonzero integers and it is partial order on $\mathbb{N}$.

For integers a, b (not both zero); $d \in \mathbb{N}$ is called a greatest common divisor (GCD) of a, b denoted by (a,b) or GCD (a,b) if

i) $d|a$, $d|b$ and
ii) for any integer e with $e|a$ and $e|b \Rightarrow e|d$

If $(a,b) = 1$ then a and b are called relatively prime (coprimes).

For integers a, b, c (not all zero); $d \in \mathbb{N}$ is called a GCD of a, b, c denoted by (a,b,c) if

i) $d|a$, $d|b$, $d|c$ and
ii) For $e \in \mathbb{Z}$; $e|a$, $e|b$ and $e|c \Rightarrow e|d$.

Similarly we define GCD of four or more integers.

For non-zero integers a,b; $\ell \in \mathbb{N}$ is a least common multiple (LCM) of a, b and denoted by $[a,b]$ if

i) $a|\ell$, $b|\ell$ and
ii) $a|m$ and $b|m$ for $m \in \mathbb{Z} \Rightarrow \ell|m$

Similarly we define LCM of three or more nonzero integers.

Note that $(a,b) = (b,a) = (|a|, |b|)$, $[a,b] = [b,a] = [|a|,|b|]$.

1.1 **Euclid's Lemma**:
For $a(\neq 0)$, b, c $\in \mathbb{Z}$; $a|bc$ and $(a,b) = 1 \Rightarrow a|c$.

1.2 If a, b are non-zero integers and d, k $\in \mathbb{N}$ then
$(ka, kb) = k(a, b)$ and $(a,b) = d$ iff $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

1.3 If a, b, c $\in \mathbb{Z}$ and $(a,b) = (a,c) = 1$ then $(a,bc) = 1$.

1.4 Let a, b $\in \mathbb{N}$ and $d = (a,b)$. Then
i) There exist p, q $\in \mathbb{N}$ with $a = pd$, $b = qd$ and $(p,q) = 1$;
ii) $\ell = pqd = [a,b]$ and
iii) $ab = \ell d$

1.5 [1] For any nonzero integers a, b, h, k; $(ah, bk) = (a, b)\,(h, k)\left(\frac{a}{(a,b)}, \frac{k}{(h,k)}\right)\left(\frac{b}{(a,b)}, \frac{h}{(h,k)}\right)$

In particular $(ah, bk) = (a, k)\,(b, h)$ if $(a, b) = (h, k) = 1$.

Proof:

Let $d = (a, b)$, $f = (k, h)$. Then there exist p, q, r, s $\in \mathbb{Z}$ with $a = pd$, $b = qd$, $h = rf$, $k = sf$ and $(p, q) = (r, s) = 1$.

Hence $(ah, bk) = (prdf, qsdf) = df\,(pr, qs)$.

Let $\alpha = (p, s)$, $\beta = (q, r)$. Then $\exists$ $p_1, s_1, q_1, r_1 \in \mathbb{Z}$ such that

$p = p_1\alpha$, $s = s_1\alpha$, $q = q_1\alpha$, $r = r_1\alpha$ and $(p_1, s_1) = (q_1, r_1) = 1$.

$\therefore (pr, qs) = (p_1r_1\,\alpha\beta, q_1s_1\alpha\beta) = \alpha\beta(p_1r_1, q_1s_1) = \alpha\beta$,

since $(p_1, q_1) = (r_1, s_1) = q$ as $(p, q) = (r, s) = 1$ and $p_1|p$, $q_1|q$, $r_1|r$, $s_1|s$.

Now $(r_1, q_1) = (p_1, s_1) = 1$ gives $(p_1r_1, q_1) = 1 = (p_1r_1, s_1)$ and hence $(p_1r_1, q_1s_1) = 1$ and (**) follows. Using (**) in (*), we get $(ah, hk) = df\,\alpha\beta = (a, b)(h, k)(p, s)(q, r)$

$$= (a, b)(h, k)\left(\frac{a}{(a,b)}, \frac{k}{(h,k)}\right)\left(\frac{b}{(a,b)}, \frac{h}{(h,k)}\right)$$

as $p = \frac{a}{d} = \frac{a}{(a,b)}$, $s = \frac{k}{f} = \frac{k}{(h,k)}$ etc.

For more details and proofs of above results, one may refer [2], [3] or any standard book on Elementary Number Theory.

2. Application 1

Now we derive a formula for requiring least number of identical rectangular tiles which need to pave a square floor without breaking any tile.

2.1 Proposition:

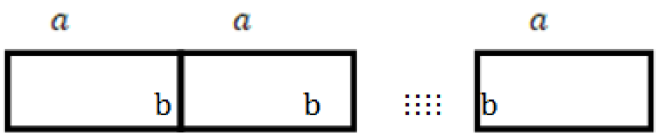
If a, b ∈ℕ and there are rectangular tiles of size

$a \times$ b (sq. units) then to form a square of smallest size by fitting these tiles requires $\frac{ab}{d^2}$ tiles, where d = $(a$,b). Moreover for any

k ∈ N, fitting $\frac{k^2ab}{d^2}$ tiles we form a square.

Proof:

Let d = $(a$,b) where a, b ∈ℕ. Then $\frac{a}{d}, \frac{b}{d} \in \mathbb{N}$. Now form a row of $\frac{b}{d}$ tiles, where each tile is of length a and width b.



$(\frac{b}{d}$tiles in this row.)

This row forms a rectangle of size $\frac{ab}{d} \times b$.

Consider $\frac{a}{d}$ such rows. Thus we have $\frac{b}{d} \times \frac{a}{d}$ tiles forming $\frac{a}{d}$ rows and $\frac{b}{d}$ columns, forming a rectangular floor and each side of this rectangle is $\frac{a\times b}{d} = \frac{b\times a}{d}$, i.e. the rectangle is a square of side $\frac{ab}{d}$ unit and number of tiles in this square floor is $\frac{b}{d} \times \frac{a}{d}$ $= \frac{ab}{d^2} = \frac{[a,b]}{(a,b)}$ by 1.4.

[Area of the square = $(\frac{ab}{d})^2 = ab \times \frac{ab}{d^2}$ = (Area of a tile).(number of tiles)].

Let X be any square formed by tiles. Let t, s be number of tiles in row and column respectively in the square X, i.e. X is formed by these tiles.

Now four sides of X are equal gives at = bs

$\Rightarrow \frac{a}{d}t = \frac{b}{d}s$ and hence $\frac{a}{d} | \frac{bs}{d}$, $\frac{b}{d} | \frac{at}{d}$

$\Rightarrow \frac{a}{d}$|s and $\frac{b}{d}\,|t$ since $(\frac{a}{d}, \frac{b}{d}) = 1$ (by 1.1)

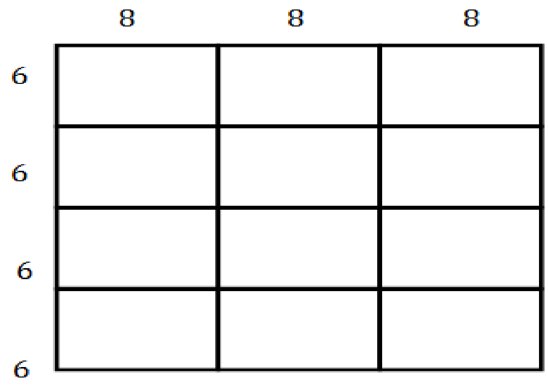
$\Rightarrow \frac{a}{d} \leq$ s and $\frac{b}{d} \leq$ t, i.e. $\frac{ab}{d^2} \leq$ st = Number of tiles in X.

This proves that the smallest square is formed with $\frac{ab}{d^2}$ tiles.

Note that such 4 squares, 9 squares, form squares.

Illustration 1: The smallest square floor which can be completely paved with tiles of size 8x6, without breaking any tile needs

$\frac{8x6}{(8,6)^2} = \frac{48}{4} = 12$ tiles.



3. Relation between LCM and GCD of three numbers.

For any non-zero integers a, b, c and k, d ∈ N, we have

$(a, b, c) = (b, c, a) = (c, a, b) = (a, c, b) = (|a|, |b|, |c|)$,

$[a, b, c] = [b, c, a] = [c, a, b] = [a, c, b] = [|a|, |b|, |c|]$,

$(ka, kb, kc) = k(a, b, c)$, $[ka, kb, kc] = k[a, b, c]$ and

$(a, b, c) = d$ iff $(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) = 1$

3.1 Theorem: Let $a, b, c \in \mathbb{N}$ and $d = (a, b, c)$, $d_1 = (\frac{a}{d}, \frac{b}{d})$, $d_2 = (\frac{b}{d}, \frac{c}{d})$, $d_3 = (\frac{a}{d}, \frac{c}{d})$. Then

i) There exist $a, b, c \in \mathbb{N}$ such that $a = dd_1d_3a_1$, $b = dd_1d_2b_1$, $c = dd_2d_3c_1$;
ii) $(d_1,d_2) = (d_2,d_3) = (d_1,d_3) = 1$
iii) $(a_1,d_2) = (b_1,d_3) = (c_1,d_1) = 1$
iv) $(a_1,b_1) = (b_1,c_1) = (a_1,c_1) = 1$
v) $[a, b, c] = dd_1d_2d_3a_1b_1c_1$ is the LCM of a, b, c.
vi) $abc = [a, b, c](a, b, c)^2 (a,b)(b,c)(a,c)$.

Proof: Let $a, b, c \in \mathbb{N}$, $d = (a, b, c)$, $d_1 = (\frac{a}{d}, \frac{b}{d})$, $d_2 = (\frac{b}{d}, \frac{c}{d})$, $d_3 = (\frac{a}{d}, \frac{c}{d})$.

ii) Let $h = (d_1, d_2)$. Then $h|d_1$, $h|d_2$ ($d_1|\frac{a}{d}$, $d_1|\frac{b}{d}$, $d_2|\frac{c}{d}$) it gives

$h|\frac{a}{d}$, $h|\frac{b}{d}$, $h|\frac{c}{d}$ and hence h divides $(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) = 1$,

i.e. $h = 1 = (d_1, d_2)$. Similarly $(d_2, d_3) = (d_1, d_3) = 1$.

i) Now $d_1|\frac{a}{d}$, $d_3|\frac{a}{d}$, and $(d_1, d_3) = 1 \Rightarrow d_1d_3|\frac{a}{d}$ by 1.3.

Hence there exists $a_1 \in \mathbb{N}$ such that $\frac{a}{d} = d_1d_3a_1$

i.e. $a = dd_1d_3a_1$. Similarly there exist $b_1, c_1 \in N$

such that $b = dd_1d_2b_1$, $c = dd_2d_3c_1$.

iii) Let $g = (a_1, d_2)$. Then $g|\frac{a}{d}$, $g|\frac{b}{d}$, $g|\frac{c}{d}$ as $a_1|\frac{a}{d}$ and $d_2 = (\frac{b}{d}, \frac{c}{d})$.

$\Rightarrow$g divides $(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}) = 1$, i.e. $(a_1, d_2) = g = 1$.

Similarly $(b_1, d_3) = (c_1, d_1) = 1$.

iv) Let $f = (a_1, b_1)$. Then $f|\frac{a}{dd_1}$, $f|\frac{b}{dd_1}$ by (i)

$\Rightarrow$ f divides $(\frac{a}{dd_1}, \frac{b}{dd_1}) = 1$, since $d_1 = (\frac{a}{d}, \frac{b}{d})$.

i.e. $(a_1, b_1) = f = 1$. Similarly $(b_1, c_1) = (a_1, c_1) = 1$.

v) $\ell = dd_1d_2d_3a_1b_1c_1 \in \mathbb{N}$ and $\ell = ad_2b_1c_1 = bd_3a_1c_1 = cd_1a_1b_1$ by (i)

$\Rightarrow$ $a|\ell$, $b|\ell$ and $c|\ell$.

Let $m \in \mathbb{Z}$ be such that $a|m$, $b|m$, $c|m$. Then by (i),

$d_1d_3a_1|\frac{m}{d}$, $d_1d_2b_1|\frac{m}{d}$, $d_2d_3c_1|\frac{m}{d}$.

Now, $(d_1, c_1) = 1 = (d_1, d_2d_3)$ by (ii) and (iii) $\Rightarrow (d_1, d_2d_3c_1) = 1$.

$\Rightarrow d_1d_2d_3c_1|\frac{m}{d}$ as $d_1|\frac{m}{d}$ and $d_2d_3c_1|\frac{m}{d}$.

Similarly $d_1d_2d_3a_1|\frac{m}{d}$, $d_1d_2d_3b_1|\frac{m}{d}$.

Thus $a_1|\frac{m}{dd_1d_2d_3}$, $b_1|\frac{m}{dd_1d_2d_3}$, $c_1|\frac{m}{dd_1d_2d_3}$.

As $(a_1, c_1) = (b_1, c_1) = 1 = (a_1, b_1) \Rightarrow (a_1b_1, c_1) = 1$, so from

above we have $a_1b_1c_1|\frac{m}{dd_1d_2d_3}$ i.e. $dd_1d_2d_3a_1b_1c_1|m$.

i.e. $\ell = dd_1d_2d_3a_1b_1c_1$ divides m.

Hence $\ell = dd_1d_2d_3a_1b_1c_1 = [a, b, c]$ is the LCM of a, b, c.

iii) By (i), $abc = d^3(d_1d_2d_3)^2a_1b_1c_1$
$= [a, b, c]d^2d_1d_2d_3$ by (v)
$= [a, b, c]\,(a, b, c)^2(a,b)(b, c)(a, c)$.■

3.2 Corollary: If $a, b, c \in \mathbb{N}$ and $d = (a, b, c)$ such that $a = da_1$,

$b = db_1$, $c = dc_1$ and $(a_1, b_1) = (b_1, c_1) = (a_1, c_1) = 1$ then

$abc = [a, b, c](a, b, c)^2$.

Proof follows from theorem 3.1, since here $d_1= d_2= d_3 = 1$.

Cuboid of size $a$xbxc is a solid rectangular parallelepiped of length a, width b and height c. (units).

3.3 Proposition:

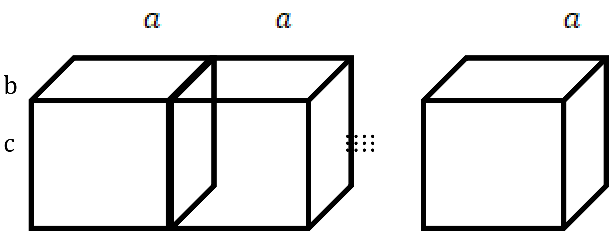
If a, b, c $\in \mathbb{N}$ and there are cuboids of same size $a$xbxc (cubic units), then using $\frac{(abc)^2}{(d^2d_1d_2d_3)^3}$ cuboids we can form a solid cube. Moreover for any k $\in \mathbb{N}$, using $\frac{k^3(abc)^2}{(d^2d_1d_2d_3)^3}$ cuboids, we can form a solid cube, where d = (a, b, c),

$$d_1 = (\frac{a}{d}, \frac{b}{d}),\ d_2 = (\frac{b}{d}, \frac{c}{d}),\ d_3 = (\frac{a}{d}, \frac{c}{d}).$$

Proof: Let a, b, c $\in \mathbb{N}$; $a = dd_1d_3a_1$, $b = dd_1d_2b_1$, $c = dd_2d_3c_1$

Where d = (a, b, c), $d_1 = (\frac{a}{d}, \frac{b}{d})$, $d_2 = (\frac{b}{d}, \frac{c}{d})$, $d_3 = (\frac{a}{d}, \frac{c}{d})$

and $a_1, b_1, c_1 \in \mathbb{N}$.



$\frac{bc}{d^2d_1d_2d_3}$ cuboids in the row.

Consider cuboids as tiles of sizes $a$xb with thickness (height) c.

Now we form a square floor with $\frac{bc}{d^2d_1d_2d_3}$ columns and $\frac{ac}{d^2d_1d_2d_3}$ rows of tiles.

[Note that $\frac{bc}{d^2d_1d_2d_3}, \frac{ac}{d^2d_1d_2d_3}, \frac{ab}{d^2d_1d_2d_3} \in \mathbb{N}$.]

Thus we have a cuboid with square base of side $\frac{abc}{d^2d_1d_2d_3}$ and height c.

Now consider such $\frac{ab}{d^2d_1d_2d_3}$ cuboids (with square base) and installing them we get a solid cube of edge $\frac{abc}{d^2d_1d_2d_3}$. For this forming the solid cube we require

$\frac{bc}{d^2d_1d_2d_3} \times \frac{ac}{d^2d_1d_2d_3} \times \frac{ab}{d^2d_1d_2d_3} = \frac{(abc)^2}{(d^2d_1d_2d_3)^3}$ cuboids.■

Remark:

In proposition 3.2, if $(d_2, c_1) = (d_3, c_1) = (d_1, b_1) = 1$.

Then number of cuboids required to form smallest cube is $\frac{(abc)^2}{(d^2d_1d_2d_3)^3}$.

Let X be a cube formed from cuboids each of size $a$xbxc. Let r, s, t be a number of cuboids along the sides of the cube X. As each edge of X is same, we have

ar = bs = ct, i.e. $d_1d_3a_1r = d_1d_2b_1s = d_2d_3c_1t$

$\Rightarrow d_3a_1r = d_2b_1s$, $d_1a_1r = d_2c_1t$

Now $d_2b_1|d_3a_1r$ with $(d_2, d_3) = (d_2, a_1) = 1$, so $(d_2, d_3a_1) = 1$,

$(b_1, a_1) = 1 = (b_1, d_3) = 1$, so $(b_1, d_3a_1) = 1$ (See theorem 3.1)

and hence $(d_2b_1, d_3a_1) = 1$, which gives by 1.1, $d_2b_1|r$.

Also we have $c_1|d_1a_1r$ and $(c_1, d_1) = (c_1, a_1) = 1$,

i.e. $(c_1, d_1a_1) = 1$, so again by 1.1, $c_1|r$.

$d_2b_1|r$, $c_1|r$ and $(b_1, c_1) = 1$, $(d_2, c_1) = 1$ i.e. $(d_2b_1, c_1) = 1$

gives $d_2b_1c_1|r$ (by 1.1). Now $\frac{bc}{d^2d_1d_2d_3} = d_2b_1c_1$ divides r.

Thus $\frac{bc}{d^2d_1d_2d_3} \le$ r.

Using $(d_3, c_1) = (d_1, b_1) = 1$, we get

$$\frac{ac}{d^2d_1d_2d_3} = d_3a_1c_1 \le s,\ \frac{ab}{d^2d_1d_2d_3} = d_1a_1b_1 \le t.$$

Hence $\frac{abc^2}{(d^2d_1d_2d_3)^3} \le$ rst = Number of cuboids in X.

This proves that the smallest cube is formed with $\frac{(abc)^2}{(d^2d_1d_2d_3)^3}$ cuboids.

If $d_1=d_2=d_3=1$ then $\frac{abc^2}{(d^2d_1d_2d_3)^3} = \frac{abc^2}{(d)^6}$

3.4 Corollary [Application 2]

Let a, b, c $\in \mathbb{N}$, d = (a, b, c), $d_1 = (\frac{a}{d}, \frac{b}{d})$, $d_2 = (\frac{b}{d}, \frac{c}{d})$, $d_3 = (\frac{a}{d}, \frac{c}{d})$ and $a = dd_1d_3a_1$, $b = dd_1d_2b_1$,

$c = dd_2d_3c_1$. If one of the following holds:

$(d_1, a_1) = 1$, $(d_1, b_1) = 1$, $(d_2, b_1) = 1$, $(d_2, c_1) = 1$, $(d_3, a_1) = 1$, $(d_3, c_1) = 1$

then using $\frac{(abc)^2}{(d^2d_1d_2d_3)^3}$ cuboids, each of size a x b x c (cubic units), we can form the smallest cube without breaking any cuboid.

Hint: In the above remark, for $(d_2, c_1) = 1$, we have $\frac{bc}{d^2d_1d_2d_3} \leq$ r, i.e., $\frac{abc}{d^2d_1d_2d_3} \leq ar$

$$\Rightarrow \left(\frac{abc}{d^2d_1d_2d_3}\right)^3 \leq (ar)^3 = \text{Volume of the cube X}$$

From this corollary 3.4 follows.

Illustration 2: Determination of the number of cuboids, each of size 100x210x375 (cubic units) to form the smallest solid cube without breaking any cube.

For $a = 100 = 5\times2 \times 5\times2$, $b = 210 = 5\times2\times3\times7$, $c = 375 = 5\times3\times5\times5 \in \mathbb{N}$.

We have $d = (a, b, c) = 5$, $d_1 = (\frac{a}{d}, \frac{b}{d}) = 2$, $d_2 = (\frac{b}{d}, \frac{c}{d}) = 3$, $d_3 = (\frac{a}{d}, \frac{c}{d}) = 5$, $a_1 = 2$, $b_1 = 7$, $c_1 = 5$, with d_1, d_2, d_3 are pairwise relatively prime and a_1, b_1, c_1 pairwise relatively prime with $(a_1, d_2) = (b_1, d_3) = (c_1, d_1) = 1$ (verifies theorem 3.1).

Note that $(d_1, a_1) = 2$, $(d_1, b_1) = 1 = (d_2, b_1) = (d_2, c_1) = (d_3, a_1) = 1$ and $(d_3, c_1) = 5$.

So by corollary 3.4; to form smallest cube from cuboids, each of size a$\times$b$\times$c (cubic units) without breaking any cuboid requires number of cuboids

$$\frac{abc}{(d^2d_1d_2d_3)^3} = \frac{100\times210\times375}{(5^2\times2\times3\times5)^3} = \frac{(7875000)^2}{(750)^3} = 147000.$$

3.5 Corollary. Let $a, b, c \in \mathbb{N}$, $d = (a, b, c)$ and $a = da_1$, $b=dc_1$, $c=dc_1$ with

$$(a_1, b_1) = (b_1, c_1) = (a_1, c_1) = 1.$$

Using $\frac{abc^2}{(d)^6}$ cuboids, each of size $a$xbxc, we can form smallest solid cube. For any $k \in \mathbb{N}$, using $\frac{k^3(abc)^2}{(d)^6}$ such cuboids we form a solid cube.

Illustration 3: To form smallest cube from cuboids each of size $10\times6\times4$, we require

$$\frac{(10\times6\times4)^2}{(10,6,4)^6} = \frac{(10\times6\times4)^2}{2^6} = (5\times3\times2)^2 = 900.$$

CONCLUSION:

Using proposition 3.1 and corollary 3.4, we can solve problems of constructing a square floor or a cube from identical tiles (cuboids)

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