A STUDY ABOUT FUNDAMENTAL GROUP IN ALGEBRAIC TOPOLOGY

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Abstract - *The research paper deals with the study of Fundamental Group with the help of examples and some of its applications. In the mathematical field of algebraic topology, the Fundamental Group records information about the basic shape, or holes, of the topological space. The concept of Fundamental Group has been detailed with the help of some theorems and lemmas. Preliminaries are being discussed before moving to the main part.*

Key Words: **Topological Space, Homotopy, Equivalence Relation, Equivalence Class, Continous Function, Group, Subspace, Isomorphism**

1. INTRODUCTION

In mathematics of algebraic topology, 'The Fundamental Group' is a mathematical group associated to any given pointed topological space that provides a way to determine when two paths, starting and ending at a fixed base point, can be continuously deformed into each other. 'The Fundamental Group' is the first and simplest homotopy group..

1.1 Paths and Loops

A **path** in a topological space X is a continuous function α from the closed unit interval I = [0,1] into X. The points α(0) and α(1) are the initial point and terminal point of α respectively. Paths α and β with common initial point α(0) = β(0) and common terminal point α(1) = β(1) are equivalent provided that there is a continuous function $H : I \times I \to X$ such that

$H(t, 0)=\alpha(t), H(t, 1)=\beta(t), t \in I,$

$H(0,s) = \alpha(0) = \beta(0), H(1,s) = \alpha(1) = \beta(1), s \in I$

The function H is called a homotopy between α and β. For a given value of s, the restriction of H to I × {s} is called s-level of the homotopy and is denoted as $H(\cdot, s)$.

A **loop** in a topological space X is a path α in X with α(0) = α(1). The common value of the initial point and terminal point is referred to as the base point of the loop. Two loops α and β having common base point x_0 are equivalent or homotopic modulo x_0 provided that they are equivalent as paths. In other words α and β are homotopic modulo x_0 (denoted $\alpha \sim_{x_0} \beta$) provided that there is a homotopy $H: I \times I \to X$ such that

$H(\cdot, 0) = \alpha, H(\cdot, 1) = \beta,$

$H(0,s) = H(1,s) = x_0, s \in I$

Theorem: Equivalence of loops is an equivalence relation on the set of loops in X with base point x_0

1.2 Path Product

If α and β are paths in X with α(1) = β(0), then the path product α∗β is the path defined by,

$$\alpha*\beta(t) = \begin{cases} \alpha(2t) & ;\text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1); & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Lemma: Suppose loops $\alpha, \alpha', \beta, \beta'$ in a space X all have base point x_0 and satisfy the relation $\alpha \sim_{x_0} \alpha'$ and $\beta \sim_{x_0} \beta'$. Then the products $\alpha * \beta$ and $\alpha' * \beta'$ are homotopic modulo x_0 .

2.1 The Fundamental Group

Consider the family of loops in X with base point x_0. Homotopy modulo x_0 is an equivalence relation on this family and therefore partitions it into disjoint equivalence classes, [α] denoting the equivalence class determined by loop α. The class [α] is called the homotopy class of α. The set of such homotopy classes is denoted by $\pi_1(X, x_0)$.If [α] and [β] belong to $\pi_1(X, x_0)$,then the product $[\alpha] \circ [\beta]$ is defined as follows:

$$[\alpha] \circ [\beta] = [\alpha * \beta].$$

Thus the product of two homotopy classes is the class determined by the path product of their representative elements. The set $\pi_1(X, x_0)$ with the ∘ operation is called the fundamental group of X at x_0 , the first homotopy group of X at x_0 .

Theorem: The set $\pi_1(X, x_0)$ is a group under the ∘ operation.

Proof: There is a constant loop c defined by

$$c(t) = x_0 , t \in I.$$

such that [c] is an identity element in $\pi_1(X, x_0)$

Also, each homotopy class $[\alpha]$ has an inverse

$[\alpha]^{-1} = [\bar{\alpha}]$, where

$$\bar{\alpha}(t) = \alpha(1 - t), \quad t \in I,$$

called the reverse of the path α

And, multiplication $\circ$ is associative as we have

$$([\alpha] \circ [\beta]) \circ [\gamma] = [\alpha] \circ ([\beta] \circ [\gamma]).$$

Hence, the set $\pi_1(X, x_0)$ is a group under the $\circ$ operation.

3. Examples of Fundamental Group

3.1 Deformation Retraction

Let X be a space and A, a subspace of X. Then, A is a deformation retract of X means that there is a homotopy $H : X \times I \to X$ such that

$$H(x, 0) = x,\ H(x,1) \in A,\ x \in X,$$

$$H(a, t) = a,\ a \in A,\ t \in I.$$

The homotopy H is called a deformation retraction.

Theorem: If A is a deformation retraction of a space X and is a point of A, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(A, x_0)$.

Example: Consider the punctured plane $\mathbb{R}^2\setminus\{p\}$consisting of all points in $\mathbb{R}^2$ expect a particular point p. Let A be a circle with center p

For $x \in \mathbb{R}^2 \setminus \{p\}$, the half line from p through x intersects the circle A at a point r(x). Define a homotopy $H: (\mathbb{R}^2\setminus\{p\}) \times I \to \mathbb{R}^2\setminus\{p\}$by

$$H(x, t) = t(x) + (1-t)x, \quad x \in \mathbb{R}^2\setminus\{p\},\ t \in I.$$

It is easy to see that H is a deformation retraction, so A is deformation retract of $\mathbb{R}^2 \setminus \{p\}$. Thus

$$\pi_1(\mathbb{R}^2\setminus\{p\}) \cong \pi_1(A) \cong Z.$$

Example: Consider an annulus X in the plane. Both the inner and outer circles of X are deformation retracts, so $\pi_1(X)$ is the group of integers.

Example: Each of the following spaces is contractible, so each has fundamental group {0}:

(a) a single point,

(b) an interval on the real line,

(c) the real line,

(d) Euclidean n-space $\mathbb{R}^n$,

(e) any convex set in $\mathbb{R}^n$

4. CONCLUSION

In this paper, we took up the study of ' The Fundamental Group'. We also studied briefly about homotopic paths, the fundamental group and the examples of fundamental group. Based on the study we arrive at a conclusion that the fundamental group is a tool from algebraic topology that can tell you what a topological space looks like.

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