ON CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS: AN APPLICATION

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Abstract - *In the present paper, we have studied a class WR(λ,β,α,μ,θ) which consist of analytic and univalent functions with negative coefficients in the open disk $U = \{z \in C : |z| < 1\}$ defined by Hadamard product with Rafid Operator,we obtain coefficient bounds , extreme points for this class ,Also weighted mean , arithmetic mean and some results.*

Key Words**: Univalent function, Rafid operator, Extreme point, Hadamard product, Weighted mean, Arithmetic mean.**

1. INTRODUCTION

Let R stand in favor of mapping

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0, n \in N = \{1,2,3,\dots\}) \qquad (1)$$

whichever analytic and univalent in the unit disk

$U = \{z \in C : |z| < 1\}$ If $f \in R$ is specified in (1) and $g \in R$ specified in

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0$$

after that effective Hadamard product f*g of f and g is clear with

$$f*g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (g*f)(z) \qquad (2)$$

Lemma 1. The Rafid Operator of $f \in R$, $0 \leq \mu < 1$, $0 \leq \theta \leq 1$ is denoted by R_μ^θ and defined as following

$$R_\mu^\theta(f(z)) = \frac{1}{(1-\mu)^{1+\theta}\,\lceil\theta+1}\int_0^\infty t^{\theta-1} e^{-\left(\frac{1}{1-\mu}\right)} f(zt)dt$$

$$= z - \sum_{n-2}^{\infty} k(n,\mu,\theta) a_n z^n \qquad (3)$$

wherever $k(n,\mu,\theta) = \frac{(1-\mu)^{n-1}\lceil\theta+n}{\lceil\theta+1}$

Proof: $R_\mu^\theta(f(z)) = \frac{1}{(1-\mu)^{1+\theta}\lceil\theta+1}\int_0^\infty t^{\theta-1} e^{-\left(\frac{1}{1-\mu}\right)} f(zt)dt$

$$= \frac{1}{(1-\mu)^{1+\theta}\lceil\theta+1}\int_0^\infty t^{\theta-1} e^{-\left(\frac{1}{1-\mu}\right)}\left[zt - \sum_{n=2}^{\infty} a_n (zt)^n\right]dt$$

$$= \frac{1}{(1-\mu)^{1+\theta}\lceil\theta+1}\left[z\int_0^\infty t^\theta e^{-\left(\frac{1}{1-\mu}\right)}dt - \sum_{n=2}^{\infty} a_n z^n \int_0^\infty t^{\theta-1+n} e^{-\left(\frac{1}{1-\mu}\right)}dt\right]$$

Thus

$$R_\mu^\theta(f(z)) = \frac{1}{(1-\mu)^{1+\theta}\lceil\theta+1}\left[\left[z\int_0^\infty (1-\mu)^{1+\theta} e^{-x} x^\theta dx\right]\right] - \frac{1}{(1-\mu)^{1+\theta}\lceil\theta+1}\left[\sum_{n=2}^{\infty} a_n z^n \int_0^\infty (1-\mu)^{\theta+n} e^{-x} x^{\theta-1+n} dx\right]$$

$$= \frac{1}{(1-\mu)^{1+\theta}\lceil\theta+1}\left[z(1-\mu)^{1+\theta}\lceil\theta+1 - \sum_{n=2}^{\infty} a_n z^n (1-\mu)^{\theta+n}\lceil\theta+n\right]$$

$$= z - \sum_{n=2}^{\infty} \frac{(1-\mu)^{n-1}\lceil\theta+n}{\lceil\theta+1} a_n z^n$$

$$= z - \sum_{n-2}^{\infty} k(n,\mu,\theta) a_n z^n$$

Definition1. A function $f(z) \in R$, $z \in U$ is said to be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$
if and only if satisfies the inequality

$$\operatorname{Re}\left\{\frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2 (R_\mu^\theta((f*g)(z)))''}{(1-\mu)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'}\right\}$$

$$\geq \beta\left|\frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2 (R_\mu^\theta((f*g)(z)))''}{(1-\mu)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'} - 1\right| + \alpha \quad (4)$$

somewhere $0 \leq \mu < 1, 0 \leq \theta \leq 1$ $0 \leq \alpha < 1, \beta \geq 0, z \epsilon U$

and g(z) are given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0$$

Lemma 2. Let w = u+iv. Then Re w$\geq \sigma$ iff

$$|w-(1+\sigma)| \leq |w+(1-\sigma)|$$

Lemma 3. Let w = u+iv and σ,γ are real numbers.

Then Re w> $\sigma|w-1|+\gamma$ if and only if

$$\operatorname{Re}\left\{w(1+\sigma e^{i\phi}) - \sigma e^{i\phi}\right\} > \gamma$$

We endeavor to study the coefficient bounds, extreme points, Hadamard product of the class WR(λ,β,α,μ,θ) , wighted mean, arithmetic can and some results.

2. COEFFICIENT BOUNDS AND EXTREME POINTS:

We acquire the essential and satisfactory circumstance and extreme points for the functions f(z) in the class WR(λ,β,α,μ,θ).

Therom2.1 The mapping f(z) clear with (1) is in the class WR(λ,β,α,μ,θ) iff

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]k(n,\mu,\theta)a_n b_n \leq 1-\alpha \quad (5)$$

wherever $0 \leq \mu < 1, 0 \leq \theta \leq 1$ $0 \leq \alpha < 1$,

$0 \leq \lambda \leq 1, \beta \geq 0$

Proof; By clarification (1),we get

$$\operatorname{Re}\left\{\frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2 (R_\mu^\theta((f*g)(z)))''}{(1-\mu)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'}\right\}$$

$$\geq \beta\left|\frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2 (R_\mu^\theta((f*g)(z)))''}{(1-\mu)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'} - 1\right| + \alpha$$

subsequently through Lemma 3, we comprise

$$\operatorname{Re}\left\{\frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2 (R_\mu^\theta((f*g)(z)))''}{(1-\mu)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'} X(1+\beta e^{i\phi}) - \beta e^{i\phi}\right\} \geq \alpha$$

$-\pi < \phi \leq \pi$,or consistently,

$$\operatorname{Re}\left\{\frac{z(R_\mu^\theta((f*g)(z)))' + \lambda z^2 (R_\mu^\theta((f*g)(z)))''(1+\beta e^{i\phi})}{(1-\mu)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'} - \frac{\beta e^{i\phi}((1-\lambda)(R_\mu^\theta((f*g)(z)) + \lambda z^2 (R_\mu^\theta((f*g)(z)'))}{(1-\mu)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'}\right\} \geq \alpha$$

(6)

Let F(z) =

$$[z(R_\mu^\theta((f*g)(z)))' + \lambda z^2 (R_\mu^\theta((f*g)(z)))''](1+\beta e^{i\phi})$$

$$-\beta e^{i\phi}[(1-\lambda)(R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))']$$

And

$$E(f) = (1-\mu)R_\mu^\theta((f*g)(z)) + \lambda z(R_\mu^\theta((f*g)(z)))'$$

next to Lemma 2. (6) is comparable to

$|F(Z)+(1-\alpha)E(Z)| \geq |F(Z)-(1+\alpha)E(Z)|$ for $0 \leq \alpha < 1$But $|F(Z)+(1-\alpha)E(Z)|$ =

$$-\beta e^{i\phi}\left[(1-\lambda)(z-\sum_{n=2}^{\infty}k(n,\mu,\theta)a_n b_n z^n)\right]$$

$$-\beta e^{i\phi}\left[\lambda z + \lambda\sum_{n=2}^{\infty}nk(n,\mu,\theta)a_n b_n z^n\right]$$

$$+(1-\alpha)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)k(n,\mu,\theta)a_n b_n z^n)\right]\Bigg|$$

$$=\left|(2-\alpha)z-\sum_{n=2}^{\infty}\left[(n+\lambda n(n-1)+(1-\alpha)(1-\lambda+n\lambda\right]k(n,\mu,\theta)a_n b_n z^n\right.$$

$$\left.-\beta e^{i\phi}\sum_{n=2}^{\infty}\left[n+n\lambda(n-1)-(1-\lambda+n\lambda)\right]k(n,\mu,\theta)a_n b_n z^n\right|$$

$$\geq(2-\alpha)|z|-\sum_{n=2}^{\infty}\left[(n+\lambda n(n-1)+(1-\alpha)(1-\lambda+n\lambda\right]k(n,\mu,\theta)a_n b_n|z|^n$$

$$-\beta\sum_{n=2}^{\infty}\left[n+\lambda n(n-2)-1+\lambda\right]k(n,\mu,\theta)a_n b_n|z|^n$$

Also $|F(Z)-(1+\alpha)E(Z)|=$

$$\left|-az-\sum_{n=2}^{\infty}\left[(n+\lambda n(n-1)-(1+\alpha)(1-\lambda+n\lambda)\right]k(n,\mu,\theta)a_n b_n z^n\right.$$

$$\left.-\beta e^{i\phi}\sum_{n=2}^{\infty}\left[n+n\lambda(n-1)-(1-\lambda+n\lambda)\right]k(n,\mu,\theta)a_n b_n z^n\right|$$

$$\leq\alpha|z|+\sum_{n=2}^{\infty}\left[(n+\lambda n(n-1)-(1+\alpha)(1-\lambda+n\lambda\right]k(n,\mu,\theta)a_n b_n|z|^n$$

$$+\beta\sum_{n=2}^{\infty}\left[n+\lambda n(n-1)-(1-\lambda+n\lambda)\right]k(n,\mu,\theta)a_n b_n|z|^n$$

Furthermore

$$|F(Z)+(1-\alpha)E(Z)|-|F(Z)-(1+\alpha)E(Z)| \geq 2(1-\alpha)|z|$$

$$\sum_{n=2}^{\infty}\begin{bmatrix}(2n+2\lambda n(n-1)-2\alpha(1-\lambda+n\lambda)\\-\beta(2n+2n\lambda(n-1)-2(1-\lambda+n\lambda))\end{bmatrix}k(n,\mu,\theta)a_n b_n|z|^n\geq 0$$

Or

$$\sum_{n=2}^{\infty}\begin{bmatrix}n(1+\beta)+n\lambda(n-1)(1+\beta)-\\(1-\lambda+n\lambda)(\beta+\alpha)\end{bmatrix}k(n,\mu,\theta)a_n b_n$$
$$\leq 1-\alpha$$

This is comparable to

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]k(n,\mu,\theta)a_n b_n\leq 1-\alpha$$

on the contrary, expect that (5) holds. afterward we obliged to show

$$\operatorname{Re}\left\{\begin{array}{l}\dfrac{z(R_\mu^\theta((f*g)(z)))'+\lambda z^2(R_\mu^\theta((f*g)(z)))''(1+\beta e^{i\phi})}{(1-\mu)R_\mu^\theta((f*g)(z))+\lambda z(R_\mu^\theta((f*g)(z)))'}-\\ \dfrac{\beta e^{i\phi}((1-\lambda)(R_\mu^\theta((f*g)(z))+\lambda z^2(R_\mu^\theta((f*g)(z)))'))}{(1-\mu)R_\mu^\theta((f*g)(z))+\lambda z(R_\mu^\theta((f*g)(z)))'}\end{array}\right\}\geq\alpha$$

3. HADAMARD PRODUCT

Theorem : $f(z)=z-\sum_{n=2}^{\infty}a_n z^n$ and $g(z)=z-\sum_{n=2}^{\infty}b_n z^n$

belong to WR(λ,β,α,μ,θ)

afterward effective Hadamard product of f and g is given

by $f*g(z)=z-\sum_{n=2}^{\infty}a_n b_n z^n=(g*f)(z)$

Proof:

Since f and g $\in WR(\lambda,\beta,\alpha,\mu,\theta)$

We have

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]k(n,\mu,\theta)b_n}{1-\alpha}\right]a_n\leq 1$$

And

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]k(n,\mu,\theta)a_n}{1-\alpha}\right]b_n\leq 1$$

and by applying the Cauchy-Schwarz ineuqality, we have

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]k(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha}\right]\sqrt{a_n b_n}$$

$$\leq\left(\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]k(n,\mu,\theta)b_n}{1-\alpha}\right]a_n\right)^{1/2}$$

$$\times\left(\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]k(n,\mu,\theta)a_n}{1-\alpha}\right]b_n\right)^{1/2}$$

Consequently we attain

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]k(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha}\right]\sqrt{a_n b_n}\leq 1$$

Now we want to prove

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]k(n,\mu,\theta)}{1-\alpha}\right]a_n b_n\leq 1$$

Since

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]\mathrm{k}(\mathrm{n},\mu,\theta)}{1-\alpha}\right]a_n b_n$$

$$= \sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]\mathrm{k}(\mathrm{n},\mu,\theta)\sqrt{a_n b_n}}{1-\alpha}\right]\sqrt{a_n b_n}$$

thus we search out the consequence.

4. WEIGHTED MEAN AND ARITHMETIC MEAN

Lemma 4.

If $\mathrm{Re}\, w \geq \gamma|w-1|+k$, where $0 \leq k \prec 1$, $\gamma \geq 0$.

Then $|w| \geq \dfrac{\gamma+k}{\gamma+1}$

Proof: Let $\mathrm{Re}\, w \geq \gamma|w-1|+k$, as $|w| \geq \mathrm{Re}$,

we acquire

$|w| \geq \gamma|w-1|+k$, or equivalent $|w|(1+\gamma) \geq \gamma+k$,

subsequently $|w| \geq \dfrac{\gamma+k}{\gamma+1}$

Defination 2. Allow f(z) and g(z) belong to R. subsequently the weighed mean $h_j(z)$ of f(z) and g(z) is given by

$$h_j(z) = \frac{1}{2}[(1-j)f(z) + (1+j)g(z)] \quad (7)$$

Definition 3. The arithmetic mean of f_j

(j = 1,2,....,q) is

clear within $W(z) = \dfrac{1}{q}\sum_{j=1}^{q} f_j(z)$ (8)

In the next theorem we will show the weighted mean

and arithmetic mean in the class

Theorem.

If f(z) and g(z) are in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$ Afterward the weighted mean defined by Definition 2 is in the class

$WR(\lambda,\beta,\alpha,\mu,\theta)$,where

$$f(z) = z-\sum_{n=2}^{\infty} c_n z^n, \quad g(z) = z-\sum_{n=2}^{\infty} d_n z^n$$

Proof: By definition 2, we attain

$h_j(z)=$

$$\frac{1}{2}\left[(1-j)\left(z-\sum_{n=2}^{\infty} c_n z^n\right) + (1+j)\left(z-\sum_{n=2}^{\infty} d_n z^n\right)\right]$$

$$= z-\sum_{n=2}^{\infty}\frac{1}{2}\left[(1-j)c_n+(1+j)d_n\right]z^n$$

We necessity explain so as to $h_j(z)$ so by lemma 2 we get

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]k(n,\mu,\theta)$$

$$X\left[\frac{1}{2}(1-j)c_n+(1+j)d_n\right]b^n$$

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]k(n,\mu,\theta)\left[\frac{1}{2}(1-j)\right]c_n b_n$$

$$+\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]k(n,\mu,\theta)\left[\frac{1}{2}(1+j)\right]d_n b_n$$

$$\leq\left[(1-j)+(1+j)\right](1-\alpha)=1-\alpha$$

The proof is complete.

Thorem: Let $f_j(z)$ clear with

$$f_j(z) = z-\sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0,\ j=1,2,\ldots\ldots q) \quad (9)$$

$$\left|\frac{z(\mathrm{R}_{\mu}^{\theta}((f*g)(z)))' + \lambda z^2(\mathrm{R}_{\mu}^{\theta}((f*g)(z)))''}{(1-\lambda)\mathrm{R}_{\mu}^{\theta}((f*g)(z))+\lambda z(\mathrm{R}_{\mu}^{\theta}((f*g)(z)))'}\right| \geq \frac{\beta+\alpha}{\beta+1}$$

Proof: Commencing (8) and (9) we container inscribe

$$W(z) = \frac{1}{q}\sum_{j=1}^{q}\left(z-\sum_{n=2}^{\infty} a_{n,j} z^n\right)$$

$$= z-\sum_{j=1}^{q}\left(\frac{1}{q}\sum_{n=2}^{\infty} a_{n,j}\right)z^n$$

because $f_j(z) \in WR(\lambda,\beta,\alpha,\mu,\theta)$ for every

(j=1,2,......q), so by using the theorem we get

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]k(n,\mu,\theta)\left[\frac{1}{q}\sum_{n=2}^{q}a_{n,j}\right]b_n$$
$$=\frac{1}{q}\sum_{n=2}^{q}\left[\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]k(n,\mu,\theta)a_{n,j}b_n\right]$$
$$\leq\frac{1}{q}\sum_{n=2}^{q}(1-\alpha)=(1-\alpha)$$

This is the absolute verification.

Theorem:

Let f(z) clear with (1) be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$ .Then

$$\left|\frac{z(R_{\mu}^{\theta}((f*g)(z)))'+\lambda z^2(R_{\mu}^{\theta}((f*g)(z)))''}{(1-\lambda)R_{\mu}^{\theta}((f*g)(z))+\lambda z(R_{\mu}^{\theta}((f*g)(z)))'}\right|\geq\frac{\beta+\alpha}{\beta+1} \qquad (10)$$

Proof: As f(z)∈ $WR(\lambda,\beta,\alpha,\mu,\theta)$ .after that by lemma 4 ,we achieve

$$\left|\frac{z(R_{\mu}^{\theta}((f*g)(z)))'+\lambda z^2(R_{\mu}^{\theta}((f*g)(z)))''}{(1-\lambda)R_{\mu}^{\theta}((f*g)(z))+\lambda z(R_{\mu}^{\theta}((f*g)(z)))'}\right|\geq\frac{\beta+\alpha}{\beta+1}$$

The verification is comprehensive.

5. CONCLUSION

Using Hadamard product with Rafid Operator, we obtained coefficient bounds, extreme points of the class WR(λ,β,α,μ,θ), Also described weighted mean, arithmetic mean and some results.

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