Stability of Collinear Libration Point L3 in Photo – Gravitational Restricted Problem of 2+2 Bodies When Bigger Primary is a Triaxial Rigid Body Perturbed by Coriolis and Centrifugal Forces

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ABSTRACT - *Stability of collinear libration point L_3 in photo – gravitational restricted problem of 2+2 bodies when bigger primary is a triaxial rigid body perturbed by coriolis and centrifugal forces has been studied in which it is found that L_3 is unstable.*

Keywords: Stability, Collinear libration point, Photo-gravitation, Triaxial rigid body, Coriolis force, Centrifugal force.

1. INTRODUCTION

Equilibrium solutions of restricted problem of 2+2 bodies are derived by Whipple (1984) in which M_1 and M_2 are two point masses moving in the circular Keplerian orbit about their centre of mass. He assumed that $M_1 \geq M_2$. Two minor bodies m_1 and m_2 $(m_1, m_2 << M_2)$ move in the gravitational fields of primaries (M_1 and M_2). They attract each other but do not perturb the primaries. He showed the existence of fourteen equilibrium solutions. Among these, six equilibrium solutions are located about the collinear Lagrangian points of classical restricted problem of three bodies and eight equilibrium solutions are found in the neighborhood of triangular Lagrangian points.

Sharma, Taqvi and Bhatnagar (2001) studied the existence and stability of libration points in the restricted three body problem when primaries are triaxial rigid body and source of radiation. Five libration points were found by them, three collinear and two triangular, they also observed that triangular points are stable while collinear libration points are unstable for the mass parameter $0 \leq \mu < \mu_{\text{crit}}$.

Garain and Chakraborty (2007) derived libration points and examined stability in Robe's restricted three body problem when second primary is a triaxial rigid body perturbed by coriolis and centrifugal forces. They found that the collinear libration points are deviated because of perturbation of centrifugal forces and triaxility of the second primary. Triaxility character of the body and perturbation of Coriolis force play an important role for finding the region of stability.

Hoque and Garain (2014) computed collinear libration point L_3. In the case of 2+2 body problem when perturbation effects act in coriolis and centrifugal forces, small primary is a radiating body and bigger primary as a triaxial rigid body.

2. EQUATIONS OF MOTION

Whipple's (1984) equation of motion of restricted problem of 2 + 2 bodies in synodic system be

$$\ddot{x}_i - 2\dot{y}_i = \frac{\partial T}{\partial x_i} \tag{1}$$

$$\ddot{y}_i + 2\dot{x}_i = \frac{\partial T}{\partial y_i} \tag{2}$$

$$\ddot{z}_i = \frac{\partial T}{\partial z_i}, \qquad (i = 1, 2) \tag{3}$$

$$T=\sum_{i=1}^{2}\mu_i\left[\frac{(x_i^2+y_i^2)}{2}+\frac{(1-\mu)}{r_{1i}}+\frac{\mu}{r_{2i}}+\frac{\mu_{3-i}}{2r}\right]$$

$$\mu=\frac{M_2}{M_1+M_2},\ \mu_i=\frac{m_i}{M_1+M_2},\ (i=1,2),\qquad r_{1i}^2=(x_i-\mu)^2+y_i^2+z_i^2,\ r_{2i}^2=(x_i-\mu+1)^2+y_i^2+z_i^2$$

$$r^2=(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2$$

Here we consider M_2, a radiating body and M_1, a triaxial rigid body, we have also considered effect of perturbation in coriolis and centrifugal forces in this configuration.

Hence force function T reduces to U as follows:

$$U=\sum_{i=1}^{2}\mu_i\left[\frac{\beta(x_i^2+y_i^2)}{2}+\frac{(1-\mu)}{r_{1i}}+\frac{q\mu}{r_{2i}}+\frac{\mu_{3-i}}{2r}+\frac{(1-\mu)(2\sigma_1-\sigma_2)}{2r_{1i}^3}-\frac{3(1-\mu)(\sigma_1-\sigma_2)y_i^2}{2r_{1i}^5}\right] \quad (4)$$

Where, $q=1-\in, \beta=1+\in'$ and $i=1,2$.

Equilibrium points of the system are those points where $\dot{x}_i=\dot{y}_i=\dot{z}_i=\frac{\partial U}{\partial x_i}=\frac{\partial U}{\partial y_i}=\frac{\partial U}{\partial z_i}=0,\ (i=1,2)$.

Thus we get

$$\beta x_1-\frac{(1-\mu)(x_1-\mu)}{r_{11}^3}-\frac{q\mu(x_1-\mu+1)}{r_{21}^3}-\frac{\mu_2(x_1-x_2)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_1-\mu)}{2r_{11}^5}$$
$$+\frac{15(1-\mu)(\sigma_1-\sigma_2)(x_1-\mu)y_1^2}{2r_{11}^7}=0 \quad (5)$$

$$\beta y_1-\frac{(1-\mu)y_1}{r_{11}^3}-\frac{q\mu y_1}{r_{21}^3}-\frac{\mu_2(y_1-y_2)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)y_1}{2r_{11}^5}$$
$$-\frac{3(1-\mu)(\sigma_1-\sigma_2)}{2}\left\{\frac{2y_1}{r_{11}^5}-\frac{5y_1^3}{r_{11}^7}\right\}=0 \quad (6)$$

$$-\frac{(1-\mu)z_1}{r_{11}^3}-\frac{q\mu z_1}{r_{21}^3}-\frac{\mu_2(z_1-z_2)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)z_1}{2r_{11}^5}+\frac{15(1-\mu)(\sigma_1-\sigma_2)y_1^2z_1}{2r_{11}^7}=0 \quad (7)$$

$$\beta x_2-\frac{(1-\mu)(x_2-\mu)}{r_{12}^3}-\frac{q\mu(x_2-\mu+1)}{r_{22}^3}-\frac{\mu_1(x_2-x_1)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_2-\mu)}{2r_{12}^5}$$
$$+\frac{15(1-\mu)(\sigma_1-\sigma_2)(x_2-\mu)y_2^2}{2r_{12}^7}=0 \quad (8)$$

$$\beta y_2-\frac{(1-\mu)y_2}{r_{12}^3}-\frac{q\mu y_2}{r_{22}^3}-\frac{\mu_1(y_2-y_1)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)y_2}{2r_{12}^5}$$
$$-\frac{3(1-\mu)(\sigma_1-\sigma_2)}{2}\left\{\frac{2y_2}{r_{12}^5}-\frac{5y_2^3}{r_{12}^7}\right\}=0 \quad (9)$$

$$-\frac{(1-\mu)z_2}{r_{12}^3}-\frac{q\mu z_2}{r_{22}^3}-\frac{\mu_1(z_2-z_1)}{r^3}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)z_2}{2r_{12}^5}+\frac{15(1-\mu)(\sigma_1-\sigma_2)y_2^2z_2}{2r_{12}^7}=0 \quad (10)$$

From (7) and (10), we get $z_1 = z_2 = 0$.

By inspection, it can be seen that equations (6) and (9) are satisfied when $y_1 = y_2 = 0$.

Now we have to determine x_1 and x_2 such that the following simplified forms of equations (5) and (8) are satisfied.

$$\therefore \beta x_1 - \frac{(1-\mu)(x_1-\mu)}{|x_1-\mu|^3} - \frac{q\mu(x_1-\mu+1)}{|x_1-\mu+1|^3} - \frac{\mu_2(x_1-x_2)}{|x_1-x_2|^3}$$

$$-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_1-\mu)}{2|x_1-\mu|^5} + \frac{15(1-\mu)(\sigma_1-\sigma_2)(x_1-\mu)y_1^2}{2|x_1-\mu|^7} = 0 \quad (11)$$

And $$\beta x_2 - \frac{(1-\mu)(x_2-\mu)}{|x_2-\mu|^3} - \frac{q\mu(x_2-\mu+1)}{|x_2-\mu+1|^3} - \frac{\mu_2(x_2-x_1)}{|x_2-x_1|^3}$$

$$-\frac{3(1-\mu)(2\sigma_1-\sigma_2)(x_2-\mu)}{2|x_2-\mu|^5} + \frac{15(1-\mu)(\sigma_1-\sigma_2)(x_2-\mu)y_2^2}{2|x_2-\mu|^7} = 0 \quad (12)$$

The solution of equations (11) and (12) can be obtained with the help of power series.

Let $\in_i = \frac{\mu_i}{(\mu_1+\mu_2)^{\frac{2}{3}}}, \ (i=1,2)$ (13)

$$\therefore \in_1 = \frac{\mu_1}{(\mu_1+\mu_2)^{\frac{2}{3}}} \text{ and } \in_2 = \frac{\mu_2}{(\mu_1+\mu_2)^{\frac{2}{3}}} \quad (14)$$

$$\therefore \mu_2 \in_1 = \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^{\frac{2}{3}}} = \mu_1 \in_2 = k(say) \quad (15)$$

Let $x_1 = L_i' + \sum_{j=1}^{n} a_{1j} \in_2^j, \text{for } i = 1,\ 2,\ 3$ (16)

Where L_1', L_2', L_3' equilibrium points are in photo-gravitational restricted problem of three bodies when bigger primary is a triaxial rigid body perturbed by coriolis and centrifugal forces and x_1 be the x coordinate of first small body.

$$x_2 = L_i' + \sum_{j=1}^{n} a_{2j} \in_1^j \text{ for } i = 1,2,3 \quad (17)$$

Similar to Whipple,

$$x_1 = L_i' + \frac{(\pm 1)}{(\Omega_{xx}^{\circ})^{\frac{1}{3}}} \frac{\mu_2}{(\mu_1+\mu_2)^{\frac{2}{3}}} \text{ where } i = 1,2,3 \quad (18)$$

And

$$x_2 = L_i' - \frac{(\pm 1)}{(\Omega_{xx}^{\circ})^{\frac{1}{3}}} \frac{\mu_1}{(\mu_1+\mu_2)^{\frac{2}{3}}} \text{ where } i = 1,2,3 \quad (19)$$

Hoque and Garain (2014) obtained two values $(x_1, 0, 0)$ and $(x_2, 0, 0)$ of L_3 in which

$x_1 = a_{31} - b_{31} \in + c_{31} \in' - 2d_{31}\sigma_1 + d_{31}\sigma_2$

And $x_2 = a_{32} - b_{32} \in + c_{32} \in' - 2d_{32}\sigma_1 + d_{32}\sigma_2$

Where, $a_{31} = \mu + a_3 + \dfrac{(\pm 1)\mu_2}{A_3^{\frac{1}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$, $b_{31} = b_3 + \dfrac{(\pm 1)\mu_2 B_3}{3A_3^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$

$c_{31} = c_3 - \dfrac{(\pm 1)\mu_2 C_3}{3A_3^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$, $d_{31} = d_3 + \dfrac{(\pm 1)\mu_2 D_3}{3A_3^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$

$a_{32} = \mu + a_3 - \dfrac{(\pm 1)\mu_1}{A_3^{\frac{1}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$, $b_{32} = b_3 - \dfrac{(\pm 1)\mu_1 B_3}{3A_3^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$

$c_{32} = c_3 + \dfrac{(\pm 1)\mu_1 C_3}{3A_3^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$ and $d_{32} = d_3 - \dfrac{(\pm 1)\mu_1 D_3}{3A_3^{\frac{4}{3}}(\mu_1+\mu_2)^{\frac{2}{3}}}$

They obtained the particular values of x_1 and x_2 for different values of μ, μ_1 and μ_2.

Our characteristic equation corresponding to the point $(x_1, 0, 0)$ is

$$f(\lambda) = \lambda^4 + \lambda^2\left(4 - \frac{U_{x_1x_1}}{\mu_1} - U_{y_1y_1}\right) + \frac{1}{\mu_1^2}\left(U_{x_1x_1}U_{y_1y_1} - U^2_{x_1y_1}\right) = 0 \quad (20)$$

$$\frac{U_{x_1x_1}}{\mu_1} = \left[\beta + \frac{2(1-\mu)}{r_{11}^3} + \frac{2q\mu}{r_{21}^3} + \frac{2\mu_2}{r^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{r_{11}^5} - \frac{45(1-\mu)(\sigma_1-\sigma_2)y_1^2}{r_{11}^7}\right]$$

$$\frac{U_{y_1y_1}}{\mu_1} = \left[\beta - \frac{(1-\mu)}{r_{11}^3} - \frac{q\mu}{r_{21}^3} - \frac{\mu_2}{r^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2r_{11}^5} + \frac{3(1-\mu)y_1^2}{r_{11}^5} + \frac{3q\mu y_1^2}{r_{21}^5}\right.$$
$$\left. + \frac{3\mu_2(y_1-y_2)^2}{r^5} + \frac{15(1-\mu)(7\sigma_1-6\sigma_2)y_1^2}{2r_{11}^7} - \frac{105(1-\mu)(\sigma_1-\sigma_2)y_1^4}{2r_{11}^9}\right]$$

$$\frac{U_{x_1y_1}}{\mu_1} = \frac{3(1-\mu)(x_1-\mu)y_1}{r_{11}^5} + \frac{3q\mu(x_1-\mu+1)y_1}{r_{21}^5} + \frac{3\mu_2(x_1-x_2)(y_1-y_2)}{r^5}$$
$$+ \frac{15(1-\mu)(2\sigma_1-\sigma_2)(x_1-\mu)y_1}{2r_{11}^7} + \frac{15(1-\mu)(\sigma_1-\sigma_2)(x_1-\mu)y_1}{r_{11}^7}$$

In this case, $y_1 = 0$ and $y_2 = 0$.

∴The above equations reduce to

$$\frac{U_{x_1x_1}}{\mu_1} = \left[\beta + \frac{2(1-\mu)}{(x_1-\mu)^3} + \frac{2q\mu}{(x_1-\mu+1)^3} + \frac{2\mu_2}{(x_1-x_2)^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(x_1-\mu)^5}\right]$$

$$\frac{U_{y_1y_1}}{\mu_1} = \left[\beta - \frac{(1-\mu)}{(x_1-\mu)^3} - \frac{q\mu}{(x_1-\mu+1)^3} - \frac{\mu_2}{(x_1-x_2)^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2(x_1-\mu)^5}\right]$$

And $\dfrac{U_{x_1y_1}}{\mu_1} = 0$

For the collinear equilibrium solutions the partial derivatives contained in equation (20) reduce to

$$\frac{U_{x_1x_1}}{\mu_1} = \left[\beta + \frac{2(1-\mu)}{|x_1-\mu|^3} + \frac{2q\mu}{|x_1-\mu+1|^3} + \frac{2\mu_2}{|x_1-x_2|^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{|x_1-\mu|^5}\right]$$

$$\frac{U_{x_1y_1}}{\mu_1}=0$$

$$\frac{U_{y_1y_1}}{\mu_1}=\left[(1+\in')-\frac{(1-\mu)}{|x_1-\mu|^3}-\frac{(1-\in)\mu}{|x_1-\mu+1|^3}-\frac{\mu_2}{|x_1-x_2|^3}-\frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2|x_1-\mu|^5}\right]$$

Let $\frac{U_{y_1y_1}}{\mu_1}=A_{31}-B_{31}\in+C_{31}\in'-D_{31}\sigma_1+E_{31}\sigma_2$

$$x_1=a_{31}-b_{31}\in+c_{31}\in'-2d_{31}\sigma_1+d_{31}\sigma_2$$

and

$$x_2=a_{32}-b_{32}\in+c_{32}\in'-2d_{32}\sigma_1+d_{32}\sigma_2$$

$$\Rightarrow\frac{U_{y_1y_1}}{\mu_1}=A_{31}-B_{31}\in+C_{31}\in'-D_{31}\sigma_1+E_{31}\sigma_2$$

Where, $A_{31}=1-\frac{(1-\mu)}{|a_{31}-\mu|^3}-\frac{\mu}{|a_{31}-\mu+1|^3}-\frac{\mu_2}{|a_{31}-a_{32}|^3}$,

$$B_{31}=\frac{3(1-\mu)b_{31}}{|a_{31}-\mu|^4}-\frac{\mu}{|a_{31}-\mu+1|^3}+\frac{3\mu b_{31}}{|a_{31}-\mu+1|^4}+\frac{3\mu_2(b_{31}-b_{32})}{|a_{31}-a_{32}|^4},$$

$$C_{31}=1+\frac{3(1-\mu)c_{31}}{|a_{31}-\mu|^4}+\frac{3\mu c_{31}}{|a_{31}-\mu+1|^4}+\frac{3\mu_2(c_{31}-c_{32})}{|a_{31}-a_{32}|^4},$$

$$D_{31}=\frac{6(1-\mu)d_{31}}{|a_{31}-\mu|^4}+\frac{6\mu d_{31}}{|a_{31}-\mu+1|^4}+\frac{6\mu_2(d_{31}-d_{32})}{|a_{31}-a_{32}|^4}+\frac{6(1-\mu)}{|a_{31}-\mu|^5}$$

and

$$E_{31}=\frac{3(1-\mu)d_{31}}{|a_{31}-\mu|^4}+\frac{3\mu d_{31}}{|a_{31}-\mu+1|^4}+\frac{3\mu_2(d_{31}-d_{32})}{|a_{31}-a_{32}|^4}+\frac{9(1-\mu)}{2|a_{31}-\mu|^5}$$

Here we see that $U_{x_1x_1}>0$ and $U_{x_1y_1}=0$. Equation (20) reduces to

$$f(\lambda)=\lambda^4+\lambda^2\left(4-\frac{1}{\mu_1}U_{x_1x_1}-U_{y_1y_1}\right)+\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}=0 \qquad (20a)$$

Case I: Let λ_1^2 and λ_2^2 be the roots of equation (20a).

Sub case (i) if λ_1^2 and λ_2^2 both are real and one of them is positive, let λ_1^2 is a positive quantity, then square root of λ_1^2 must be real and of opposite sign. In this case characteristic roots will be a real number. So in this case L_3 is unstable. We may get similar result in the case of λ_2^2.

Sub case (ii) if λ_1^2 and λ_2^2 both are real and negative, let λ_1^2 is a negative quantity, then two roots of λ_1^2 be purely imaginary. Similarly two roots of λ_2^2 be purely imaginary. In this case L_3 should be stable. Again when λ_1^2 and λ_2^2 both are negative quantity then

$\lambda_1^2\lambda_2^2$ = a positive quantity (20b)

From equation (20a), we have obtained that

$\lambda_1^2\lambda_2^2=\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}$ = a negative quantity $(\because U_{x_1x_1}>0,\ U_{y_1y_1}<0)$ (20c)

Since (20b) and (20c) contradict each other. So L_3 is unstable.

Now let $4-\frac{U_{x_1x_1}}{\mu_1}-U_{y_1y_1}=A_{33}$

Case II: If $A_{33}>0$ i.e., $4>\frac{1}{\mu_1}U_{x_1x_1}+U_{y_1y_1}$ then $f(\lambda)=\lambda^4+A_{33}\lambda^2+\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}=0$.

Sub case (i) it is clear that $U_{x_1x_1}>0$ and let $U_{y_1y_1}>0$ then $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}>0$ so, $f(\lambda)=0$ has no change of signs and as such it has no positive real roots. $f(-\lambda)=\lambda^4+A_{33}\lambda^2+\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}=0$ also has no change of signs and as such it has no positive real roots i.e., $f(\lambda)=0$ has no negative real roots. So in this case we can say that all roots of $f(\lambda)=0$ are imaginary.

Sub case (ii) if $U_{y_1y_1}<0$ then similar to Yadav's case $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}<0$.

Let $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}=-B_{33},\ B_{33}>0$, so, $f(\lambda)=\lambda^4+A_{33}\lambda^2-B_{33}=0$. Here $f(\lambda)=0$ has only one change of sign and as such it has one positive real root. $f(-\lambda)=\lambda^4+A_{33}\lambda^2-B_{33}=0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

Case III: If $A_{33}<0$ i.e., $4<\frac{1}{\mu_1}U_{x_1x_1}+U_{y_1y_1}$ then let $A_{33}=-C_{33}, C_{33}>0$ then

$$f(\lambda)=\lambda^4-C_{33}\lambda^2+\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}=0$$

Sub case (i) let $U_{y_1y_1}>0$, then $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}>0$

Then, $f(\lambda)=0$ has two changes of signs and as such it has at most two positive real roots. $f(-\lambda)=\lambda^4-C_{33}\lambda^2+\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}=0$ also has two change of signs and as such it has at most two positive real roots i.e., $f(\lambda)$ has at most two negative real roots.

Sub case (ii) if $U_{y_1y_1}<0$, then $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}<0$.

Let $\frac{1}{\mu_1^2}U_{x_1x_1}U_{y_1y_1}=-D_{33}, \mathrm{D}_{33}>0$. Here $f(\lambda)=\lambda^4-C_{33}\lambda^2-D_{33}=0$ has only one change of sign and as such it has one positive real root. $f(-\lambda)=\lambda^4-C_{33}\lambda^2-D_{33}=0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)=0$ has one negative real root.

In both the cases we see that $f(\lambda)=0$ has one positive and one negative real root. So the libration point L_3 is unstable.

The characteristic equation corresponding to the point $(x_2,0,0)$ be

$$f(\lambda)=\lambda^4+\lambda^2\left(4-\frac{1}{\mu_2}U_{x_2x_2}-U_{y_2y_2}\right)+\frac{1}{\mu_2^2}\left(U_{x_2x_2}U_{y_2y_2}-U_{x_2y_2}^2\right)=0 \quad (21)$$

$$\frac{U_{x_2x_2}}{\mu_2} == \left[\beta + \frac{2(1-\mu)}{(x_2-\mu)^3} + \frac{2q\mu}{(x_2-\mu+1)^3} + \frac{2\mu_1}{(x_1-x_2)^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{(x_2-\mu)^5}\right]$$

$$\frac{U_{y_2y_2}}{\mu_2} = \left[\beta - \frac{(1-\mu)}{(x_2-\mu)^3} - \frac{q\mu}{(x_2-\mu+1)^3} - \frac{\mu_1}{(x_1-x_2)^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2(x_2-\mu)^5}\right]$$

$$\frac{U_{x_2y_2}}{\mu_2} = 0$$

$$\frac{U_{x_2x_2}}{\mu_2} = \left[\beta + \frac{2(1-\mu)}{|x_2-\mu|^3} + \frac{2q\mu}{|x_2-\mu+1|^3} + \frac{2\mu_1}{|x_1-x_2|^3} + \frac{6(1-\mu)(2\sigma_1-\sigma_2)}{|x_2-\mu|^5}\right]$$

$$\frac{U_{x_2y_2}}{\mu_2} = 0$$

$$\frac{U_{x_2y_2}}{\mu_2} = \left[(1+\in') - \frac{(1-\mu)}{|x_2-\mu|^3} - \frac{(1-\in)\mu}{|x_2-\mu+1|^3} - \frac{\mu_1}{|x_1-x_2|^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2|x_2-\mu|^5}\right]$$

Let $\frac{U_{y_2y_2}}{\mu_2} = A_{32} - B_{32}\in + C_{32}\in' - D_{32}\sigma_1 + E_{32}\sigma_2$

$x_1 = a_{31} - b_{31}\in + c_{31}\in' - 2d_{31}\sigma_1 + d_{31}\sigma_2$

and

$x_2 = a_{32} - b_{32}\in + c_{32}\in' - 2d_{32}\sigma_1 + d_{32}\sigma_2$

Where, $A_{32} = 1 - \frac{(1-\mu)}{|a_{32}-\mu|^3} - \frac{\mu}{|a_{32}-\mu+1|^3} - \frac{\mu_1}{|a_{32}-a_{31}|^3}$,

$$B_{32} = \frac{3(1-\mu)b_{32}}{|a_{32}-\mu|^4} - \frac{\mu}{|a_{32}-\mu+1|^3} + \frac{3\mu b_{32}}{|a_{32}-\mu+1|^4} + \frac{3\mu_1(b_{32}-b_{31})}{|a_{32}-a_{31}|^4},$$

$$C_{32} = 1 + \frac{3(1-\mu)c_{32}}{|a_{32}-\mu|^4} + \frac{3\mu c_{32}}{|a_{32}-\mu+1|^4} + \frac{3\mu_1(c_{32}-c_{31})}{|a_{32}-a_{31}|^4},$$

$$D_{32} = \frac{6(1-\mu)d_{32}}{|a_{32}-\mu|^4} + \frac{6\mu d_{32}}{|a_{32}-\mu+1|^4} + \frac{6\mu_1(d_{32}-d_{31})}{|a_{32}-a_{31}|^4} + \frac{6(1-\mu)}{|a_{32}-\mu|^5}$$

And

$$E_{32} = \frac{3(1-\mu)d_{32}}{|a_{32}-\mu|^4} + \frac{3\mu d_{32}}{|a_{32}-\mu+1|^4} + \frac{3\mu_1(d_{32}-d_{31})}{|a_{32}-a_{31}|^4} + \frac{9(1-\mu)}{2|a_{32}-\mu|^5}$$

Here we see that $U_{x_2x_2} > 0$ and $U_{x_2y_2} = 0$

Equation (21) reduces to

$$f(\lambda) = \lambda^4 + \lambda^2\left(4 - \frac{1}{\mu_2}U_{x_2x_2} - U_{y_2y_2}\right) + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0 \qquad (21a)$$

Case I: Let λ_1^2 and λ_2^2 be the roots of equation (21a).

Sub case (i) if λ_1^2 and λ_2^2 both are real and one of them is positive, let λ_1^2 is a positive quantity, then square root of λ_1^2 must be real and of opposite sign. In this case characteristic roots will be a real number. So in this case L_3 is unstable. We may get similar result in the case of λ_2^2.

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Sub case (ii) if λ_1^2 and λ_2^2 both are real and negative, let λ_1^2 is a negative quantity, then two roots of λ_1^2 be purely imaginary. Similarly two roots of λ_2^2 be purely imaginary. In this case L_3 should be stable. Again when λ_1^2 and λ_2^2 both are negative quantity then

$\lambda_1^2\lambda_2^2$ = a positive quantity (21b)

From equation (21a), we have obtained that

$\lambda_1^2\lambda_2^2 = \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2}$ = a negative quantity $\left(\because U_{x_2x_2} > 0,\ \ U_{y_2y_2} < 0\right)$ (21c)

Since (21b) and (21c) contradict each other. So L_3 is unstable.

Now let $4 - \frac{1}{\mu_2}U_{x_2x_2} - U_{y_2y_2} = A_{34}$

Case I: If $A_{34} > 0$ i.e., $4 > \frac{1}{\mu_2}U_{x_2x_2} + U_{y_2y_2}$.

Sub case (i) let $U_{y_2y_2} > 0$ then $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} > 0$ so, $f(\lambda) = \lambda^4 + A_{34}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$ has no change of signs and as such it has no positive real roots. Also $f(-\lambda) = \lambda^4 + A_{34}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$ has no change of signs and as such it has no positive real roots i.e., $f(\lambda)$ has no negative real roots. So in this case we can say that all roots of $f(\lambda) = 0$ are imaginary.

Sub case (ii) if $U_{y_2y_2} < 0$ then similar to Yadav's case $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} < 0$.

Let $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = -B_{34}, B_{34} > 0,$ so $f(\lambda) = \lambda^4 + A_{34}\lambda^2 - B_{34} = 0$. Here $f(\lambda) = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 + A_{34}\lambda^2 - B_{34} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda)$ has one negative real root.

Case II: If $A_{34} < 0$ i.e., $4 < \frac{1}{\mu_2}U_{x_2x_2} + U_{y_2y_2}$ then let $A_{34} = -C_{34}, C_{34} > 0$ then

$f(\lambda) = \lambda^4 - C_{34}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$

Sub case (i) let $U_{y_2y_2} > 0$, then $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} > 0$. Then, $f(\lambda) = 0$ has two changes of signs and as such it has at most two positive real roots. Also $f(-\lambda) = \lambda^4 - C_{34}\lambda^2 + \frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = 0$ has two change of signs and as such it has at most two positive real roots i.e., $f(\lambda)$ has at most two negative real roots.

Sub case (ii) if $U_{y_2y_2} < 0$, then $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} < 0$.

Let $\frac{1}{\mu_2^2}U_{x_2x_2}U_{y_2y_2} = -D_{34},\ D_{34} > 0$. Here $f(\lambda) = \lambda^4 - C_{34}\lambda^2 - D_{34} = 0$ has only one change of sign and as such it has one positive real root. $f(-\lambda) = \lambda^4 - C_{34}\lambda^2 - D_{34} = 0$ has only one change of sign and as such it has one positive real root i.e., $f(\lambda) = 0$ has one negative real root.

In both cases we see that $f(\lambda)=0$ has one positive and one negative real root.

3. CONCLUSION

Hence, we get that the libration point L_3 is unstable.

Table 1. For Stability of Collinear Equilibrium Solutions L_3

$$\left(\mu_1=\mu_2=10^{-10},\ \frac{1}{\mu_1}U_{y_1y_1}=A_{31}-B_{31}\in+C_{31}\in'-D_{31}\sigma_1+E_{31}\sigma_2\right)$$

μ	A_{31}	B_{31}	C_{31}	D_{31}	E_{31}
0.00001	-1.4994095191150000	0.0000024985703897	0.4999824581478970	11.9950566507850000	7.4960374145812300
0.00002	-1.4994270120531400	0.0000049972365604	0.5000324429357930	11.9953815109582000	7.4962285606185600
0.00003	-1.4994445051493000	0.0000074959985145	0.5000824291200320	11.9957063790173000	7.4964197112541800
0.00004	-1.4994619984084100	0.0000099948562549	0.5001324166984590	11.9960312549888000	7.4966108665013400
0.00005	-1.4994794918255300	0.0000124938097842	0.5001824056732970	11.9963561388467000	7.4968020263470100
0.00006	-1.4994969854056100	0.0000149928591049	0.5002323960423900	11.9966810306173000	7.4969931908044600
0.00007	-1.4995144791424700	0.0000174920042199	0.5002823878085100	11.9970059302681000	7.4971843598573900
0.00008	-1.4995319730422900	0.0000199912451317	0.5003323809689520	11.9973308378321000	7.4973755335223200
0.00009	-1.4995494671026000	0.0000224905818429	0.5003823755248470	11.9976557532963000	7.4975667117928200
0.0001	-1.4995669613221600	0.0000249900143564	0.5004323714767730	11.9979806766545000	7.4977578946657100
0.0002	-1.4997419123363300	0.0000499896091952	0.5009324077590280	12.0012303447607000	7.4996699767099000
0.0003	-1.4999168793820800	0.0000749987871771	0.5014325836352940	12.0044808030383000	7.5015825194023600
0.0004	-1.5000918624603800	0.0001000175509631	0.5019328991394420	12.0077320516900000	7.5034955228609600
0.0005	-1.5002668615697700	0.0001250459032150	0.5024333543064440	12.0109840909054000	7.5054089871970500
0.0006	-1.5004418767075200	0.0001500838465951	0.5029339491718240	12.0142369208674000	7.5073229125186900
0.0007	-1.5006169078795500	0.0001751313837667	0.5034346837672880	12.0174905418052000	7.5092372989570300
0.0008	-1.5007919550868400	0.0002001885173933	0.5039355581267420	12.0207449539215000	7.5111521466300700
0.0009	-1.5009670183205100	0.0002252552501384	0.5044365722884540	12.0240001573665000	7.5130674556294800
0.001	-1.5011420975914100	0.0002503315846677	0.5049377262819780	12.0272561523960000	7.5149832260997100
0.002	-1.5028937720403700	0.0005016236147284	0.5099569646645970	12.0598596817206000	7.5341663362934000
0.003	-1.5046470497671500	0.0007538787587968	0.5149902242610250	12.0925425849661000	7.5533957195917900
0.004	-1.5064019308145800	0.0010070996914352	0.5200375395167100	12.1253050609318000	7.5726714920576000
0.005	-1.5081584152384500	0.0012612890931734	0.5250989449413170	12.1581473090201000	7.5919937700968200
0.006	-1.5099165030678600	0.0015164496504979	0.5301744751261390	12.1910695290250000	7.6113626703531400
0.007	-1.5116761943349100	0.0017725840558630	0.5352641647311620	12.2240719212926000	7.6307783097882600
0.008	-1.5134374890548200	0.0020296950076975	0.5403680484938520	12.2571546866156000	7.6502408056290800
0.009	-1.5152003872208700	0.0022877852103880	0.5454861612310790	12.2903180262059000	7.6697502753544200
0.01	-1.5169648888341200	0.0025468573743098	0.5506185378266240	12.3235621418580000	7.6893068367762700
0.02	-1.5346980846680800	0.0051921874387827	0.6027345449808050	12.6604907096850000	7.8874886098145600
0.03	-1.5525915315890400	0.0079387394782532	0.6563158726184090	13.0057223929241000	8.0905112291594400
0.04	-1.5706450274858800	0.0107893227421968	0.7113986523498100	13.3594669599953000	8.2984972001508000
0.05	-1.5888582418487400	0.0137468065071293	0.7680197750860020	13.7219391160852000	8.5115719326372000
0.06	-1.6072307067227600	0.0168141199977438	0.8262168937936910	14.0933585565380000	8.7298637736532000
0.07	-1.6257618070483300	0.0199942521502230	0.8860284247744380	14.4739500130229000	8.9535040359651600
0.08	-1.6444507704352300	0.0232902511996477	0.9474935472630300	14.8639432919599000	9.1826270221416100
0.09	-1.6632966562523500	0.0267052240714568	1.0106522011709900	15.2635733036084000	9.4173700432561300
0.1	-1.6822983440408000	0.0302423355553427	1.0755450827460200	15.6730800808660000	9.6578734316419100
0.2	-1.8804820266566400	0.0730656523121904	1.8293282580560100	20.3678077674160000	12.4127459549468000
0.3	-2.0910818209073700	0.1319261012929010	2.8062084800647300	26.3501206973341000	15.9192686607445000
0.4	-2.3070287960738100	0.2103364516299970	4.0540353984692600	33.9217596328318000	20.3546256951936000
0.5	-2.5132087944685400	0.3104452478954830	5.6099974178147000	43.3387694960737000	25.8699614715437000

Table 2. For Stability of Collinear Equilibrium Solutions L_3

$$\left(\mu_1 = \mu_2 = 10^{-10}, \frac{1}{\mu_2} U_{y_2 y_2} = A_{32} - B_{32} \in + C_{32} \in' - D_{32}\sigma_1 + E_{32}\sigma_2\right)$$

μ	A_{32}	B_{32}	C_{32}	D_{32}	E_{32}
0.00001	-1.5006259742950700	0.0000000014791544	1.5000676521756100	0.0052744366802804	1.5041874795169100
0.00002	-1.5006434818477100	0.0000000029583640	1.5000676694976500	0.0052745885503871	1.5042163415198700
0.00003	-1.5006609895585700	0.0000000044376288	1.5000676868192500	0.0052747404370672	1.5042452041883500
0.00004	-1.5006784974325900	0.0000000059169487	1.5000677041425900	0.0052748923140422	1.5042740675092300
0.00005	-1.5006960054648300	0.0000000073963239	1.5000677214654900	0.0052750442075640	1.5043029314956700
0.00006	-1.5007135136602400	0.0000000088757542	1.5000677387901500	0.0052751960913708	1.5043317961345300
0.00007	-1.5007310220126500	0.0000000103552397	1.5000677561138200	0.0052753479983050	1.5043606614422700
0.00008	-1.5007485305282100	0.0000000118347804	1.5000677734392300	0.0052754998954940	1.5043895274024600
0.00009	-1.5007660392044700	0.0000000133143763	1.5000677907653100	0.0052756517960937	1.5044183940216900
0.0001	-1.5007835480402000	0.0000000147940274	1.5000678080914900	0.0052758037066534	1.5044472613032600
0.0002	-1.5009586452275000	0.0000000295935744	1.5000679813857300	0.0052773230441963	1.5047359703932500
0.0003	-1.5011337584672300	0.0000000443986427	1.5000681547379000	0.0052788428149571	1.5050247454538600
0.0004	-1.5013088877604000	0.0000000592092340	1.5000683281484400	0.0052803630142453	1.5053135864993600
0.0005	-1.5014840331055200	0.0000000740253503	1.5000685016166600	0.0052818836505333	1.5056024935506200
0.0006	-1.5016591944998800	0.0000000888469934	1.5000686751413500	0.0052834047388810	1.5058914666318200
0.0007	-1.5018343719494000	0.0000001036741647	1.5000688487251100	0.0052849262482528	1.5061805057440700
0.0008	-1.5020095654550600	0.0000001185068660	1.5000690223683600	0.0052864481739272	1.5064696109016600
0.0009	-1.5021847750079900	0.0000001333450998	1.5000691960671500	0.0052879705639111	1.5067587821452500
0.001	-1.5023600006190300	0.0000001481888668	1.5000693698262700	0.0052894933607757	1.5070480194627700
0.002	-1.5041131396189000	0.0000002969312920	1.5000711105963500	0.0053047454646373	1.5099440312495500
0.003	-1.5058678839886800	0.0000004462290586	1.5000728571891400	0.0053200411799681	1.5128466702417100
0.004	-1.5076242337742400	0.0000005960839637	1.5000746096163500	0.0053353807201049	1.5157559532689300
0.005	-1.5093821890343500	0.0000007464978018	1.5000763679001000	0.0053507641724151	1.5186718971399300
0.006	-1.5111417498011600	0.0000008974723794	1.5000781320556200	0.0053661917103014	1.5215945187485100
0.007	-1.5129029161097800	0.0000010490095062	1.5000799021041800	0.0053816634335231	1.5245238349937900
0.008	-1.5146656879784400	0.0000012011109921	1.5000816780644200	0.0053971794745227	1.5274598628334600
0.009	-1.5164300654034700	0.0000013537786666	1.5000834599504500	0.0054127400258768	1.5304026192975700
0.01	-1.5181960483889500	0.0000015070143502	1.5000852477844200	0.0054283451786423	1.5333521214077700
0.02	-1.5359441748404200	0.0000030710180758	1.5001034572834000	0.0055868815122286	1.5632219340866000
0.03	-1.5538527670426200	0.0000046938770498	1.5001222821437500	0.0057500356185685	1.5937853811613100
0.04	-1.5719216259314900	0.0000063775178041	1.5001417417624200	0.0059179541375229	1.6250602218709200
0.05	-1.5901504240528800	0.0000081239294160	1.5001618561562000	0.0060907887746984	1.6570646563583500
0.06	-1.6085386965128600	0.0000099351654318	1.5001826459873000	0.0062686963633940	1.6898173317620400
0.07	-1.6270858313141000	0.0000118133458377	1.5002041325762200	0.0064518391734669	1.7233373479136400
0.08	-1.6457910591258800	0.0000137606590103	1.5002263379275500	0.0066403849655323	1.7576442623061100
0.09	-1.6646534423691700	0.0000157793638010	1.5002492847467300	0.0068345072588123	1.7927580944720100
0.1	-1.6836718636246600	0.0000178717915739	1.5002729964613900	0.0070343855169446	1.8286993295193500
0.2	-1.8820365101554500	0.0000434198335198	1.5005578959462600	0.0093938800774716	2.2384600207965200
0.3	-2.0928438251243300	0.0000795244215421	1.5009512352042000	0.0125785368412377	2.7568557950509200
0.4	-2.3090260736992200	0.0001300229163061	1.5014914898639100	0.0168988773144072	3.4104941410605900
0.5	-2.5154661315821000	0.0001997219546912	1.5022269552839500	0.0227627220830975	4.2228828164516000

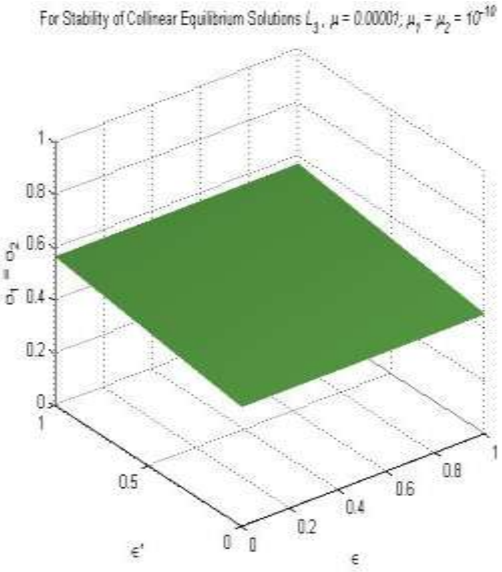


Figure 1

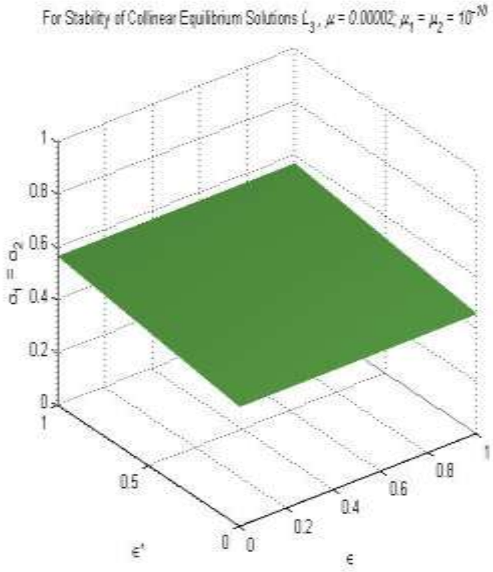
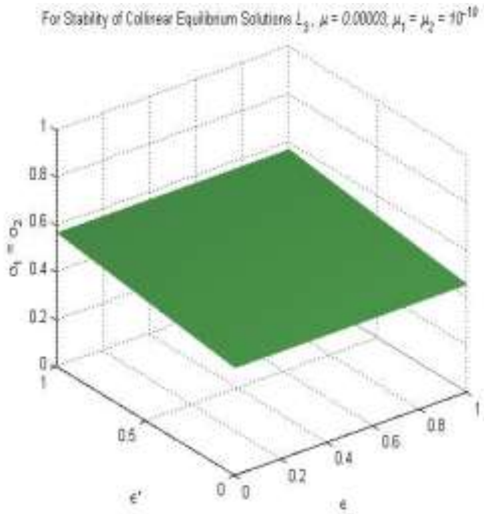


Figure 2



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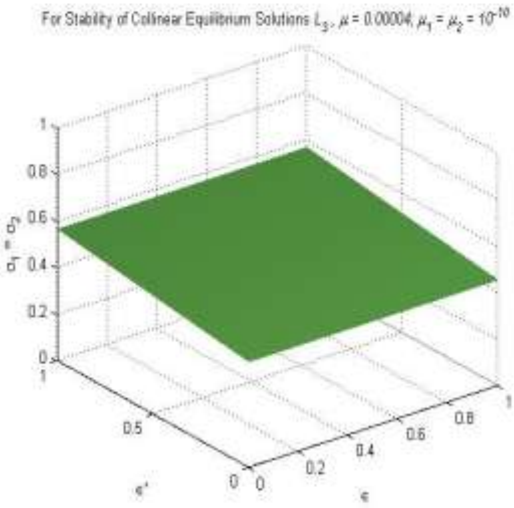

Figure 4

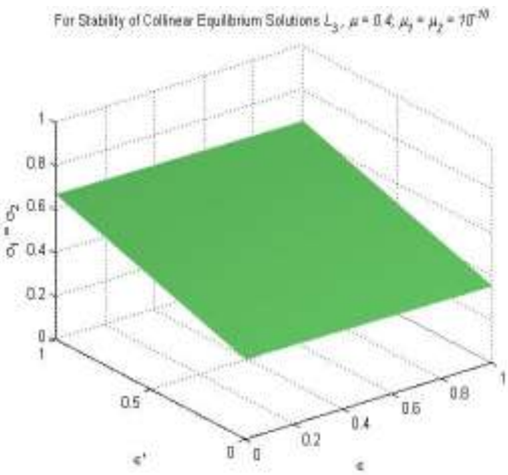

Figure 5

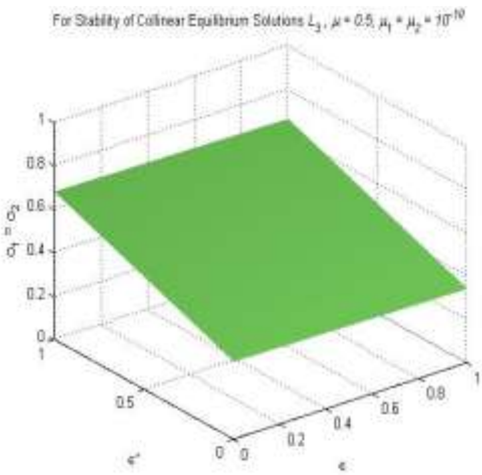

Figure 6

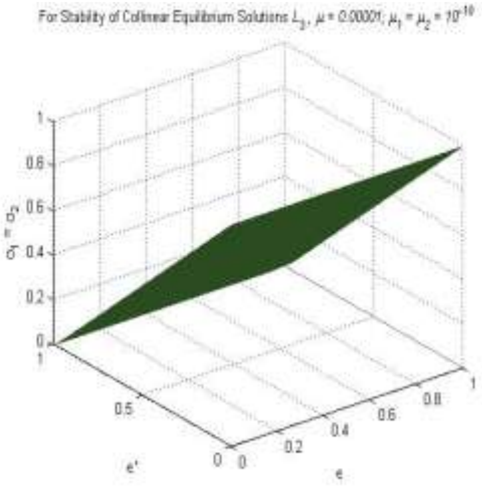

Figure 7

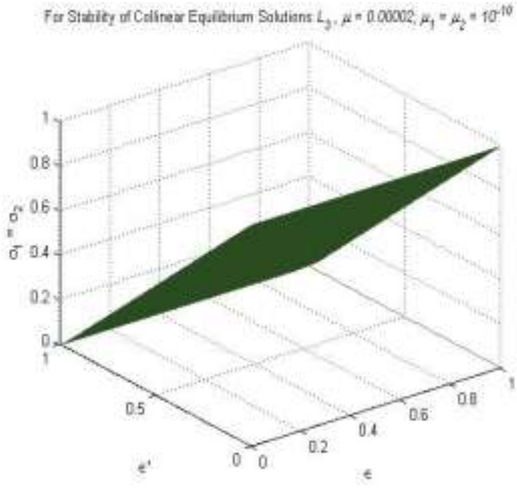

Figure 8

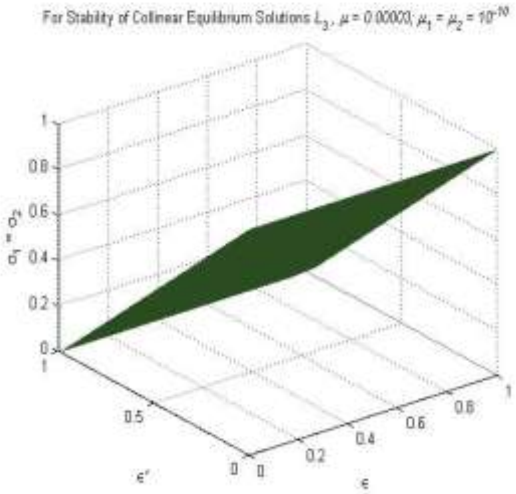

Figure 9

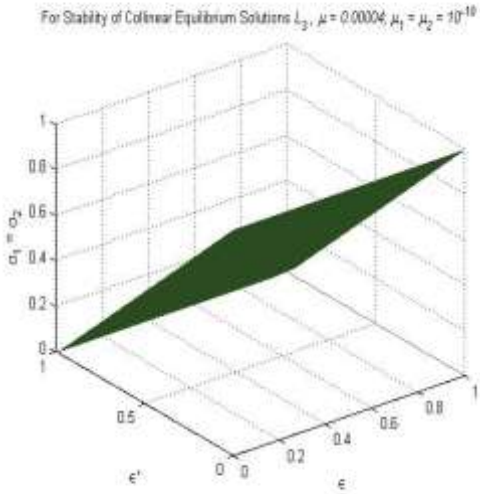

Figure 10

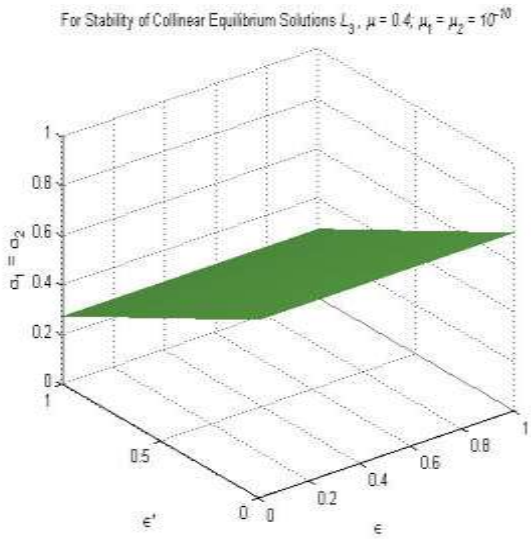

Figure 11

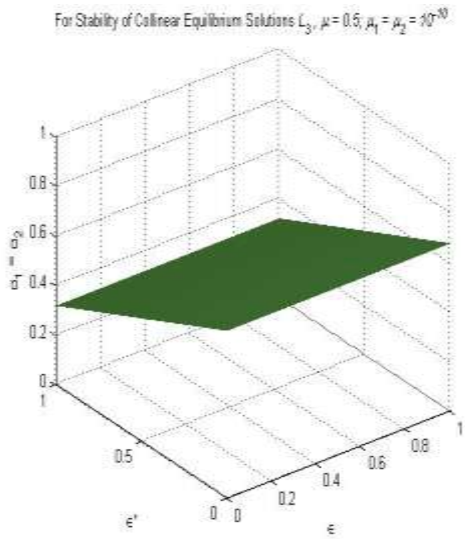

Figure 12

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BIOGRAPHY

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