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Some Note on Elimination Theory

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Abstract - *In this paper, we will study systematic methods for eliminating variable from system of polynomial equation. The main strategy of this elimination theory will be given in example. If we solve a polynomial in one variable and more than one variable, then the solution of these equations very easy way to solve, one can use a numerical approach to estimate the solutions.*

Key Words: Resultant, Elimination

1. INTRODUCTION

Many problems in linear algebra and other branches of science to solving a system of linear equations in a number of variables. This in turn means finding common solution to some polynomial equation of degree one. We are faced with non-linear system of polynomial equation in more than one variable. Elimination theory is most important for both algorithmic and complexity aspect of polynomial system solving. It also impacts several other areas of mathematics like numerical analysis, complexity, linear algebra etc. It is general about eliminating a number of unknows from a system of polynomial equations in one (or) more variables to get an equivalent system. The importance of elimination theory, let us start by considering the following example.

2. Theorem: Let us consider a system of two quadratic equations in one variable x

$$f(x) = a_1x^2+b_1x+c_1$$

$$g(x) = a_2x^2+b_2x+c_2$$

To find a necessary and sufficient condition for the existence of a common solution for the system

$$f(x) = 0$$

$$g(x) = 0$$

Proof: If $f(x)$ and $g(x)$ have a common solution, then they have a common linear factor say L.

Let $q_1(x) = f(x) / L$, $q_2(x) = g(x) / L$

Then both $q_1(x)$ & $q_2(x)$ must be linear.

We write it as $q_1(x) = A_1 x + B_1$

$$q_2(x) = - A_2 x - B_2$$

Here the signs choosen in $q_2(x)$ will make more sense in moment for some constants A_1, B_1, A_2 and B_2

Now,

$$f(x) /q_1(x) = g(x)/q_2(x) =L$$

$$f(x)\, q_2(x) = g(x)\, q_2(x)$$

Usually,

$$(a_1x^2 +b_1x^2+c_1) (-A_2x -B_2) = (a_1x^2+b_2x+c_2) (A_1X + B_1)$$

Now expanding the terms and collecting in the above equation given

$(a_1A_2+a_2A_1)\, x^3 + (b_2A_1+b_1A_2+a_1B_2+a\, B_1)\, x^2 + (c_1A_2+c_2A_1+b_2B_1+b_1B_2)\, x + (c_1B_2 + c_2B_1) = 0$

In this above equation is only possible if the coefficients of x, x^2, x^3 and the constant term are all equal to '*0*'. Now we write the following homogeneous system with variables (A_1, B_2, A_1, B_1) are arranged in this order.

$a_1A_2+a_2A_1=0$

$b_2A_1+b_1A_2+a_1B_2+a\, B_1=0$

$c_1A_2+c_2A_1+b_2B_1+b_1B_2=0$

$c_1B_2 + c_2B_1 =0$

In this above system to have a non-trivial solution, its coefficient matrix must be non – invertible. It's determinant must be zero.

$$\begin{vmatrix} a_1 & 0 & a_2 & 0 \\ b_1 & a_1 & b_2 & a_2 \\ c_1 & b_1 & c_2 & b_2 \\ 0 & c_1 & 0 & c_2 \end{vmatrix} = 0$$

Which is similar to $\begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ 0 & a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_2 & b_2 & c_2 \end{vmatrix} = 0$

Here the determinant of the matrix is equal to the determine of its transpose. This determinant is called as the resultant of f*(x)* and *g(x)*. In this case the determinant of a *4X 4* consisting of the coefficient of the two polynomials together with 0 's arranged in a way.

Definition 2.1: Let $f(x) = a_m x^m + a_{m-1}x^{m-1} + ... + a_1 x + a_0$, $g(x) = b_n x^n + b_{n-1}x^{n-1} + ... + b_1 x + b_0$ be two polynomials of degree *m* and *n* respectively such that $a_m \neq 0$ or $b_n \neq 0$. If $m \leq n$, we define the resultant of *f(x)* and *g(x)* to be following determinant

$$\text{Res}\,(f(x), g(x)) = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 \end{vmatrix}$$

We Notice that Res (*f(x), g(x)*) is the determinate of a square matrix of size $m + n$. For example, If $f(x) = -2x^4+4x^3-x^2+5x-1$ and $g(x) = x^2-6x+9$ then

$$\operatorname{Res}(f(x), g(x)) = \begin{vmatrix} 1 & -6 & 9 & 0 & 0 & 0 \\ 0 & 1 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -6 & 9 & 0 \\ 0 & 0 & 0 & 1 & -6 & 9 \\ -2 & 4 & -1 & 5 & -1 & 0 \\ 0 & -2 & 4 & -1 & 5 & -1 \end{vmatrix}$$

As a generalization of the above definition.

Example: Without solving the polynomial equation, show that the following system $x^3-3x^2+5x-3=0$; $2x^2-7x+5=0$ has solutions.

SOLUTION:

We compute the resultant of two polynomials $f(x) = x^3-3x^2+5x-3$, $g(x) = 2x^2-7x+5$

$$\operatorname{Res}(f(x), g(x)) = \begin{vmatrix} 2 & -7 & 5 & 0 & 0 \\ 0 & 2 & -7 & 5 & 0 \\ 0 & 0 & 2 & -7 & 5 \\ 1 & -3 & 5 & -3 & 0 \\ 0 & 1 & -3 & 5 & -3 \end{vmatrix}$$

Therefore the polynomials f(x), g(x) have a common root by the above theorem.

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