International Research Journal of Engineering and Technology (IRJET) e-ISSN: 2395-0056
Volume: 05 Issue: 04 | Apr-2018 www.irjet.net p-ISSN: 2395-0072

Classification of Dynkin diagrams & imaginary roots of Quasi hyperbolic Kac Moody algebras of rank 9

A Uma Maheswari

Associate Professor & Head, Department of Mathematics, Quaid-E-Millath Government College for Women, (Affiliated to University of Madras) Chennai, Tamilnadu, India

---***---

Abstract - *The theory of Kac Moody algebras has interesting applications to different branches of Mathematics and Mathematical Physics. The study of indefinite Kac-Moody algebras is one of the challenging areas in pure mathematical research. The main objective of this work is to study of rank 9 Dynkin diagrams associated with the indefinite , quasi hyperbolic class of Kac Moody algebras of rank 9. The general classification of connected, non isomorphic Dynkin diagrams associated with these rank 9 algebras is given and a study on the imaginary root system is undertaken. Computations of the bilinear forms and the Weyl group actions are given for specific classes in the rank 9 family.*

Key Words: Dynkin diagram , Kac Moody algebra, imaginary roots, fundamental reflection, indefinite, quasi hyperbolic Kac Moody algebras.

AMS MSC 17B67

1. INTRODUCTION

Kac Moody algebras, developed simultaneously by Kac and Moody around 1967 ([6],[11]), are classified into three main classes namely finite, affine and indefinite types; Among the indefinite class, extended hyperbolic type was introduced by Sthanumoorthy and Uma Maheswari in [12] while obtaining a new subclass of imaginary roots called purely imaginary roots; The quasi finite family was introduced in [5] ; Subclasses of the indefinite type, namely, the quasi hyperbolic classes and quasi affine family ([17] – [19]) were introduced by Uma Maheswari and Indefinite quasi affine Kac Moody algebras QHG_2, $QHA_2^{(1)}$,$QAC^{(1)}{}_2$ and $QAD^{(2)}{}_3$ were studied in [20] - [22];

Indefinite Kac-Moody algebras of special linear and classical type were studied in ([1], [2]) by Benkart et.al . Strictly imaginary roots and special imaginary roots were studied by Casperson [4] and Bennett [3]. In [7] – [10], study on the structure and root multiplicities for indefinite families. Extended hyperbolic Kac-Moody algebras $EHA_1^{(1)}$ and $EHA_2^{(2)}$ ([12] – [16]). Uma Maheswari studied the quasi affine family $QAG^{(1)}{}_2$;. Uma Maheswari introduced another new class of Dynkin diagrams and associated Kac Moody algebras of quasi hyperbolic type in [18]; Rank 3 Dynkin diagrams of quasi hyperbolic Kac Moody algebras were classified in [18] and properties of roots were studied.

In this paper, the main focus is on the Dynkin diagrams associated with rank 9 quasi hyperbolic Kac Moody algebras; The classification theorem which characterizes connected, non isomorphic Dynkin diagrams associated with the Generalized Cartan Matrices of indefinite, Quasi hyperbolic Kac Moody algebras of rank 9 is proved in Section 3. The imaginary roots in specific families of rank 9 are discussed and the isotropic roots are identified, and the properties of root system, namely the purely imaginary and strictly imaginary roots are discussed.

2. PRELIMINARIES

The basic definitions and concepts of Kac-Moody algebras can be referred from Kac[6] and Wan[23].

Definition 2.1[6]: An integer matrix $A=(a_{ij})_{i,j=1}^{n}$ is called as a Generalized Cartan Matrix (GCM in short) if it satisfies the following conditions:

(i) $a_{ii} = 2 \ \forall \ i = 1,2,\ldots,n$
(ii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \ \forall \ i, j = 1,2,\ldots,n$
(iii) $a_{ij} \leq 0 \ \forall \ i, j = 1,2,\ldots,n.$

Let the index set of A be denoted by N = {1,…,n}.

Definition 2.2[6]: A realization of a matrix $A=(a_{ij})_{i,j=1}^{n}$ is a triple (H, Π, Π^{v}) where l is the rank of A, H is a 2n - l dimensional complex vector space, $\Pi = \{\alpha_1,\ldots,\alpha_n\}$ and $\Pi^{v} = \{\alpha_1^{v},\ldots,\alpha_n^{v}\}$ are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^{v}) = a_{ij}$ for i, j = 1,….,n. Π is called the root basis. Elements of Π are simple roots. The Kac-Moody algebra g(A) associated with a GCM $A=(a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i , f_i , $i=1,2,..n$ and H with the following defining relations :

$$[h,h']=0, h,h' \in H, [e_i, f_j] = \delta_{ij}\alpha_i^{v}, [h,e_j] = \alpha_j(h)e_j, [h,f_j] = \alpha_j(h)f_j,$$
$$(ade_i)^{1-a_{ij}} e_j = 0, (adf_i)^{1-a_{ij}} f_j = 0 \ , \forall \ i \neq j, \ i, j \in N$$

Then we have the root space decomposition

$$g(A) = \underset{\alpha \in Q}{\oplus} g_{\alpha}(A) \text{ where}$$
$$g_{\alpha}(A) = \{x \in g(A) / [h,x] = \alpha(h)x, \ for \ all \ \ h \in H\}.$$

An element α , $\alpha \neq 0$ in Q is called a root if $g_{\alpha} \neq 0$. Let $Q_{+} = \sum_{i=1}^{n} z\alpha_i$, Q has a partial ordering "$\leq$" defined by $\alpha \leq \beta$ if $\beta - \alpha \in Q_{+}$, where $\alpha , \beta \in Q$.

Let $\Delta = (= \Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

A root α is called real, if there exists a $w \varepsilon W$ such that w(α) is a simple root;

A root which is not real is called an imaginary root.

An imaginary root α is called isotropic if (α,α) = 0.

A positive imaginary root α is a minimal imaginary root (MI root, for short) if α is minimal with respect to the partial order on H^*.

Definition 2.3[6]: The Dynkin diagram associated with the GCM A of order n, denoted by S(A) is defined as follows: S(A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|,|a_{ji}|\}$ number of lines if $a_{ij}.\, a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If $a_{ij}.\, a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}|, |a_{ji}|)$ of integers.

Definition 2.5[17]: Let A= $(a_{ij})_{i,j=1}^{n}$, be an indecomposable GCM of indefinite type. We define The associated Dynkin diagram S (A) is said to be of Quasi Hyperbolic (QH) type if S (A) has a proper connected sub diagram of hyperbolic type with n-1 vertices.

The GCM A is of Quasi Hyperbolic type if the corresponding Dynkin diagram S (A) is of QH type.

We then say the Kac Moody algebra g(A) is of Quasi Hyperbolic type.

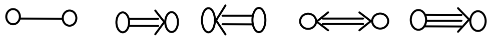
3. CLASSIFICATION THEOREM OF DYNKIN DIAGRAMS ASSOCIATED WITH QUASI HYPERBOLIC KAC MOODY ALGEBRAS OF RANK 9

In this section we shall completely classify the non isomorphic connected Dynkin diagrams of rank 9 associated with the quasi hyperbolic class.

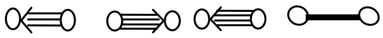
Theorem 3.1 (Classification Theorem) : There are 5 x $\sum_{i=1}^{8}(9^i\, 8C_i)$ non-isomorphic connected Dynkin diagrams associated with family of Quasi hyperbolic Kac Moody algebras of rank 9.

Proof : There are five different classes of hyperbolic Dynkin diagrams of rank 8([23]. A rank 9 quasi hyperbolic Dynkin diagram is obtained by extending one more vertex to any of the existing hyperbolic Dynkin diagrams of rank 8.

Case I) Consider the GCM associated with the Dynkin diagram of rank 8 hyperbolic diagram $H_1^{(8)}$ (notation as in Wan [27]). Extend the Dynkin diagram with an added 9th vertex which can be connected with either one , two, three, four , five, six, seven, eight vertices of the hyperbolic Dynkin diagram through 9 possible edges :



--(3.1)



Sub Case 1) α_9 is connected with only one of the 8 vertices of $H_1^{(8)}$.
This vertex can be selected from the 8 vertices in $8C_1$ ways. α_i (i=1,…,8) and α_9, can be joined by one of the 9 possible edges listed in Eqn(3.1).
Thus we obtain 9 x $8C_1$ possible Dynkin diagrams .

Sub Case 2) α_9 is connected with any two of the 8 vertices of $H_1^{(8)}$ that can be chosen from the 8 vertices in $8C_2$ ways. Thus, there are 9^2 x $8C_2$ possible Dynkin diagrams.

Sub Case 3) α_9 is connected with any three of the 8 vertices of $H_1^{(8)}$ which can be selected in $8C_3$ ways. Hence there are 9^3x $8C_3$ possible Dynkin diagrams in this case.

Sub Case 4) α_9 is connected with any four of the 8 vertices of $H_1^{(8)}$ which can be chosen in $8C_4$ ways. Thus are 9^4 x $8C_4$ possible Dynkin diagrams.

Sub Case 5) α_9 is connected with any five of the 8 vertices of $H_1^{(8)}$ that can be selected from the 8 vertices in $8C_5$ ways. There will be 9^5 x $9C_5$ possible Dynkin diagrams.

Sub Case 6) α_9 is connected with any six of the 8 vertices of $H_1^{(8)}$ which can be chosen in $8C_6$ ways. Thus, there are 9^6 x $8C_6$ Dynkin diagrams in this case.

Sub Case 7) α_9 is connected with any seven of the 8 vertices of $H_1^{(8)}$ which can be selected in $8C_7$ ways. Thus, there will be 9^7 x $8C_7$ possible Dynkin diagrams.

Sub Case 8) α_9 is connected with all the 8 vertices of $H_1^{(9)}$. Hence there will be 9^8 x $8C_8$ possible Dynkin diagrams.

In all, there are 9 x $8C_1$ + 9^2 x $8C_2$ + 9^3 x $8C_3$ + 9^4 x $8C_4$+ 9^5 x $8C_5$ + 9^6 x $8C_6$ + 9^7 x $8C_7$ + 9^8 x $8C_8$ Dynkin diagrams associated with the Quasi hyperbolic class of rank 9.

Case II) There are 5 different classes of Dynkin diagrams of rank 9 namely $H_1^{(9)}$, $H_2^{(9)}$, $H_3^{(9)}$ and $H_4^{(9)}$. In each case , there are 9 x $8C_1$ + 9^2 x $8C_2$ + 9^3 x $8C_3$ + 9^4 x $8C_4$+ 9^5 x $8C_5$ + 9^6 x $8C_6$ + 9^7 x $8C_7$ + 9^8 x $8C_8$ Dynkin diagrams as in Case I) . Thus, totally, there are

5 (9 x $8C_1$ + 9^2 x $8C_2$ + 9^3 x $8C_3$ + 9^4 x $8C_4$+ 9^5 x $8C_5$ + 9^6 x $8C_6$ + 9^7 x $8C_7$ + 9^8 x $8C_8$)

= 5 x $\sum_{i=1}^{8}(9^i\, 8C_i)$ connected Dynkin diagrams of rank 9 in the quasi hyperbolic family, proving the theorem .

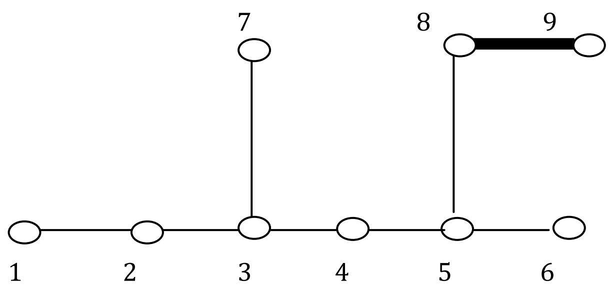
3.2 A SPECIFIC FAMILY IN THE CLASS OF QUASI HYPERBOLIC KAC-MOODY ALGEBRA $QH_1^{(9)}$

In this section , we consider a particular class of quasi hyperbolic Kac Moody algebra in the family $QH_1^{(9)}$. For simplicity of notation, let us represent this quasi hyperbolic Kac-Moody algebra by $QH_1^{(9)}$ whose associated Generalized Cartan Matrix is, is given by

$$A = (a_{ij})_{i,j=1}^{9} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & -p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p & 2 \end{pmatrix},$$

where p is any positive integer (>2).

The corresponding Dynkin diagram is



Since the GCM is symmetric, the existence of the non degenerate, symmetric bilinear form
(,) is guaranteed.
By definition, $<\alpha_i,\alpha_j>= a_{ij}$ and $(\alpha_i,\alpha_j)=2$ for all i=1,...,9.
All simple roots have the same length.
$(\alpha_1,\alpha_2) = (\alpha_2,\alpha_3)= (\alpha_3,\alpha_4)=(\alpha_4,\alpha_5)=(\alpha_5,\alpha_6)=(\alpha_3,\alpha_7)= (\alpha_5,\alpha_8)= -1$ and $(\alpha_8,\alpha_9)= -p$;

Consider roots of Height 2:

$(\alpha_1+\alpha_2, \alpha_1+\alpha_2) =2 > 0$, $\alpha_1+\alpha_2$ is a real root.
$\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$, $\alpha_3+\alpha_4$, $\alpha_4+\alpha_5$, $\alpha_5+\alpha_6$, $\alpha_3+\alpha_7$, $\alpha_5+\alpha_8$ are all real roots , since $(\alpha_1+\alpha_2, \alpha_1+\alpha_2)$, $(\alpha_2+\alpha_3, \alpha_2+\alpha_3)$, $(\alpha_3+\alpha_4, \alpha_3+\alpha_4)$, $(\alpha_4+\alpha_5, \alpha_4+\alpha_5)$, $(\alpha_5+\alpha_6, \alpha_5+\alpha_6)$, $(\alpha_3+\alpha_7, \alpha_3+\alpha_7)$, $(\alpha_5+\alpha_8,\alpha_5+\alpha_8)$ are all > 0.
$\alpha_8+\alpha_9$ is real if $4-2p >0$, (i.e). $p <2$
$\alpha_8+\alpha_9$ is imaginary if $p > 2$ and in particular, $\alpha_8+\alpha_9$ is minimal imaginary if $p> 2$
$\alpha_8+\alpha_9$ is isotropic if $p = 2$

Consider Height 3 Roots:

$(\alpha_1+ \alpha_2+ \alpha_3, \alpha_1+ \alpha_2+ \alpha_3) > 0$, hence $\alpha_1+ \alpha_2+ \alpha_3$ is a real root. Also from the bilinear form computation, we get that $\alpha_2+ \alpha_3+ \alpha_4$, $\alpha_3+ \alpha_4+ \alpha_5$, $\alpha_4+ \alpha_5+ \alpha_6$, $\alpha_2+ \alpha_3+ \alpha_7$, $\alpha_3+ \alpha_4+ \alpha_7$, $\alpha_4+ \alpha_5+ \alpha_8$, $\alpha_5+ \alpha_6+ \alpha_8$ are all real roots.

$\alpha_5+ \alpha_8+ \alpha_9$ is imaginary if $p > 2$;
$\alpha_5+ \alpha_8+ \alpha_9$ is isotropic if $p = 2$;
$\alpha_5+ \alpha_8+ \alpha_9$ is imaginary if $p > 2$

For $p >2$, $2\alpha_8+ \alpha_9$ is an imaginary root for $(2\alpha_8+ \alpha_9, 2\alpha_8+ \alpha_9) < 0$; Similarly $2\alpha_8+ \alpha_9$ is imaginary if $p > 2$ and isotropic if $p=2$.

Proposition 3.3: The Purely Imaginary Property (PIM) is satisfied by the rank 9 quasi hyperbolic Kac Moody algebra $QH_1^{(9)}$.

Proof : The Support of any root must be connected. From the Dynkin diagram of $QH_1^{(9)}$ it is clear that any imaginary root must involves $\alpha_8+ \alpha_9$.

If we take any two positive imaginary roots, then each of these roots and their addition also will involve $\alpha_8+ \alpha_9$ (in multiples of the coefficients) . Thus the addition of these two positive imaginary roots will also be an imaginary root.

Therefore the purely imaginary property is satisfied by $QH_1^{(9)}$.

Example 3.4:

Let $p >2$. Consider the two imaginary roots $\alpha = \alpha_8+ \alpha_9$ and $\beta = \alpha_5+ \alpha_8+ \alpha_9$.

Then $(\alpha+\beta, \alpha+\beta) < 0$, since $p > 2$. Hence $\alpha+\beta$ is also an imaginary root, guaranteeing the existence of purely imaginary roots.

Proposition 3.5: The quasi hyperbolic Kac Moody algebra $QH_1^{(9)}$ does not satisfy the special imaginary property.

The family $QH_1^{(9)}$ contains indefinite subclasses . By the characterization of the special imaginary roots, any Kac Moody algebra which contains indefinite subclass do not possess the special imaginary property and hence special imaginary property of roots do not hold for $QH_1^{(9)}$

Proposition 3.6: The Strictly Imaginary Property (SIM) is not satisfied for the family $QH_1^{(9)}$.

Consider the imaginary root $\alpha = \alpha_5+ \alpha_9+ \alpha_{10}$ and the real root $\beta = \alpha_2+ \alpha_3$.

It is easily seen that neither α+β nor α-β are roots since the support of α+β and support of α-β are not connected. Therefore the imaginary root α is not strictly imaginary and $\mathbf{QH_1^{(9)}}$ does not satisfy the SIM property.

CONCLUSION

The study on the rank 9 quasi hyperbolic family is undertaken here and complete classification of connected non isomorphic Dynkin diagrams of rank 9, associated with these indefinite Quasi hyperbolic Kac Moody algebras is given. Basic properties of roots are analysed. Further, this work can be extended and using the representation theory, the indepth structure of this quasi hyperbolic algebras.

ACKNOWLEDGEMENT

The author sincerely acknowledges the Tamilnadu State Council for Higher Education , Tamilnadu, for sanctioning the Minor Research Project in 2017 and this research work is part of the TANSCHE Minor Research Project Scheme.

REFERENCES

[1] Benkart, G.M., Kang, S.J., and Misra, K.C., "Indefinite Kac-Moody Algebras of Special Linear Type", Pacific Journal of Mathematics, Vol.170, No.2, (1995), pp. 379-404.

[2] Benkart, G.M., Kang, S.J., anand Misra, K.C., "Indefinite Kac-Moody Algebras of classical type", Adv. Math., Vol. 105, (1994), pp. 76-110.

[3] Bennett, C., "Imaginary roots of Kac Moody algebra whose reflections preserve root multiplicities", J.Algebra,Vol. 158, (1993), pp. 244-267.

[4] Casperson, D., "Strictly imaginary roots of Kac Moody algebras", J. Algebra, Vol. 168, (1994), pp. 90-122.

[5] Hechun Zhang, Kaiming Zhao, "Root Systems of a class of Kac Moody Algebras", Algebra Colloq. 1, no.4, (1994), 359-368.

[6] Kac, V.G. (1990), "Infinite Dimensional Lie Algebra", 3rd ed. Cambridge: Cambridge University Press. (1990)

[7] S.J. Kang, "Kac-Moody Lie algebras, spectral sequences, and the Witt formula", Trans. Amer. Math Soc., 339 (1993), 463-495.

[8] S.J. Kang, "Root multiplicities of the hyperbolic Kac-Moody algebra $HA^{(1)}{}_1$", J. Algebra, 160, (1993), 492-593.

[9] Kang, S.J., "Root multiplicities of the hyperbolic Kac-Moody Lie algebra HAn(1) ," J. Algebra, Vol. 170, (1994), pp. 277-299.

[10] S.J. Kang, "On the hyperbolic Kac-Moody Lie algebra HA(1)1" , Trans. Amer.Math. Soc., 341, (1994), 623-638.

[11] R.V. Moody, A new class of Lie algebras, J. Algebra, 10, (1968), 211-230.

[12] Sthanumoorthy, N., and Uma Maheswari, A., "Purely imaginary roots of Kac-Moody algebras", Comm. Algebra, Vol. 24, No. 2, (1996b), pp. 677-693.

[13] Sthanumoorthy, N., and Uma Maheswari, A., "Root multiplicities of extended hyperbolic Kac-Moody algebras", Comm. Algebra, Vol. 24, No. 14, (1996a), pp. 4495-4512.

[14] Sthanumoorthy, N., Lilly, P.L., and Uma Maheswari, A., "Root multiplicities of some classes of extended-hyperbolic Kac-Moody and extended - hyperbolic generalized Kac-Moody algebras", Contemporary Mathematics, AMS, Vol. 343, (2004), pp. 315-347.

[15] Sthanumoorthy, N., Uma Maheswari, A., and Lilly, P.L., "Extended – Hyperbolic Kac-Moody $EHA_2^{(2)}$ Algebras structure and Root Multiplicities", Comm. Algebra, Vol. 32, No. 6, (2004), pp. 2457-2476.

[16] Sthanumoorthy, N., and Uma Maheswari, A., "Structure And Root Multiplicities for Two classes of Extended HyberbolicKac-Moody Algebras $EHA_1^{(1)}$ and EHA2(2) for all cases", Comm. Algebra, Vol. 40, (2012), pp. 632-665.

[17] Uma Maheswari, A., "Imaginary Roots and Dynkin Diagrams of Quasi HyberbolicKac-Moody Algebras of rank 3", IJMCAR,, Vol. 4, Issue 2, (2014), pp. 19-28.

[18] Uma Maheswari. A, Quasi-Affine Kac-Moody Algebras: Classification of Dynkin Diagrams $QAG^{(1)}{}_2$, International Journal of Mathematical Sciences, Vol.34, Issue.2, (2014) 1590-1605.

[19] UmaMaheswari. A, Structure of Indefinite Quasi-Affine Kac-Moody algebras $QAD_3^{(2)}$, International Journal of Pure and Applied Mathematics, Vol 102, (2015), No.1 57-72.

[20] UmaMaheswari. A, A Study on Generalized Generalized Quasi hyperbolic Kac-Moody algebra of rank 10, Global Journal of Mathematical Sciences: Theory and Practice , Vol 9, Number 3 (2017), pp 412 – 422

[21] Uma Maheswari, A., and Krishnaveni, S., "A Study on the Structure of a class of Indefinite Non- Hyperbolic Kac-Moody Algebras QHG_2", IJMCAR, Vol. 4, Issue 4, (2014), 97-110.

[22] Uma Maheswari, A., and Krishnaveni, S., "On the Structure of Indefinite Quasi-Hyperbolic Kac-Moody Algebras $QHA_2^{(1)}$ ", Int.J. of Algebra, , 8, No.11 , (2014), 517 – 528.

[23] Wan Zhe-Xian, "Introduction to Kac-Moody Algebra", Singapore: World Scientific Publishing Co.Pvt.Ltd, (1991).