Common Fixed Point Results in Menger Spaces

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Abstract - *The purpose of the paper is to prove a common fixed point theorem in Menger spaces by using five compatible mappings.*

Key Words: Menger space, t-norm, Common fixed point, Compatible maps, Weak - compatible maps.

I. Introduction:

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by K. Menger [10] in 1942. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. Schweizer and Sklar [16] studied this concept and gave some fundamental results on this space.

The important development of fixed point theory in Menger spaces were due to Sehgal and Bharucha-Reid [13]. Sessa [14] introduced weakly commuting maps in metric spaces. Jungck [7] enlarged this concept to compatible maps. The notion of compatible maps in Menger spaces has been introduced by Mishra [11]. Recently, Singh and Jain [15] generalized the results of Mishra [11] using the concept of weak compatibility and compatibility of pair of self maps. In this paper, using the idea of weak compatibility due to Singh and Jain [15] and the idea of compatibility due to Mishra [11]. In this paper we prove a common fixed point theorem in Menger spaces by using five compatible mappings.

II. PRELIMINARIES

In [16], introduced the concept of probabilistic metric space by using the notion of triangular norm which is followings

Definition 2.1:- A triangular norm $*$ (shortly t− norm) is a binary operation on the unit interval $[0,1]$ such that for all $a, b, c, d \in [0,1]$ the following conditions are satisfied:

$(a)\ a * 1 = a;$
$(b)\ a * b = b * a;$
$(c)\ a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d;$
$(d)\ a * (b * c) = (a * b) * c.$

Example 2.2:- Two typical examples of continuous t-norm are

(a) $a * b = \max\{a + b - 1, 0\}$ *and*
(b) $a * b = \min\{a, b$

Definition 2.3:- A distribution function is a function $F: (-\infty, \infty) \rightarrow [0,1]$ *which is left continuous on* R, *non-decreasing and* $F(-\infty) = 0, F(\infty) = 1.$

We will denote by Δ the family of all distribution functions on $[-\infty, \infty]$. H is a special element of Δ defined by

$$H(t) = \begin{cases} 0 \ \ if\ t \leq 0, \\ 1 \ \ if\ t > 0. \end{cases}$$

If X is a nonempty set, $F : X \times X \rightarrow \Delta$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by $F_{x,y.}$

Definition 2.4 [16]:- The ordered pair (X, F) is called a probabilistic metric space (shortly PM-space) if X is a nonempty set and F is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0,$

$(FM-0)\ F_{x,y}\ (t) = 1 \Leftrightarrow x = y;$
$(FM-1)\ F_{x,y}\ (0) = 0,$ if t=0;
$(FM-2)\ F_{x,y} = F_{y,x};$
$(FM-3)\ F_{x,y}\ (t) = 1, F_{z,y}\ (s) = 1 \Rightarrow F_{x,z}\ (t + s) = 1.$

The ordered triple $(X, F, *)$ is called Menger space if (X, F) is a PM-space, $*$ is a t-norm and the following condition is also satisfies: for all $x, y, z \in X$ and $t, s > 0,$
(FM-4) $F_{x,y}\ (t + s) \geq F_{x,z}\ (t) * F_{z,y}(s).$

Proposition 2.5 [13]:- Let (X, d) be a metric space. Then the metric d induces a distribution function F defined by

$$F_{x,y}\ (t) = H\ (t - d(x, y))$$

for all $x, y \in X$ and $t > 0$. If t-norm $*$ is defined $a * b = min\{a, b\}$ for $all\ a, b \in [0,1]$ then $(X, F,*)$ is a Menger space. Further, $(X, F,*)$ is a complete Menger space if (X, d) is complete.

Definition 2.6 [11]:- Let $(X, F,*)$ be a Menger space and $*$ be a continuous t-norm.

(a) A sequence $\{x_n\}$ in X is said to be converge to a point x in X (written $x_n \rightarrow x$) iff for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.
(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n,\ x_{n+p}}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$.

(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.7:- If $*$ is a continuous t-norm, it follows from $(FM-4)$ that the limit of sequence in Menger space is uniquely determined.

Definition 2.8[15]:- Self maps A and B of a Menger space $(X,F,*)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Definition 2.9[11]:- Self maps A and B of a Menger space $(X,F,*)$ are said to be compatible if $F_{ABx_m,BAx_n,}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to x$, $Bx_n \to x$ for some x in X as $n \to \infty$.

Remark 2.10:- If self maps A and B of a Menger space $(X,F,*)$ are compatible then they are weakly compatible.

The following is an example of pair of self maps in a Menger space which are weakly compatible but not compatible.

Example 2.11:- Let (X,d) be a metric space where $X = [0,2]$ and $(X,F,*)$ be the induced Menger space with $F_{x,y}$ (t) = H(t – d(x, y)), $\forall x,y \in X$ and $\forall\, t > 0$. Define self maps A and B as follows:

$$Ax = \begin{cases} 2-x, & if\ 0 \le x < 1, \\ 2 & if\ 1 \le x \le 2, \end{cases}$$

and

$$Bx = \begin{cases} x, & if\ 0 \le x < 1, \\ 2, & if\ 1 \le x \le 2, \end{cases}$$

Take x_n = 1–1/n. Then $F_{Ax_{n+1,}}(t)$ = H (t– (1/n)) and $\lim n \to \infty^{F_{Ax_{n+1,}}(t)}$ = H (t) = 1. Hence $Ax_n \to \infty$ as n → ∞. Similarly, $Bx_n \to \infty$ as $n \to \infty$. Also F_{ABx_n,BAx_n} (t) = H (t – (1 – 1/n)) and lim $n \to \infty^{F_{ABx_n,BAx_n,}(t) \to 1}$ = H(t − 1) ≠ 1, $\forall t > 0$. Hence the pair (A, B) is not compatible. Set of coincidence points of A and B is $[1,2]$. Now for any $x \in [1,2]$, $Ax = Bx = 2$, and $AB(x) = A(2) = 2 = S(2) = SA(x)$. Thus A and B are weakly compatible but not compatible.

Lemma 2.12:- Let $\{x_n\}$ be a sequence in a Menger space $(X,F,*)$ with continuous t-norm $*$ and $t * t \ge t$. If there exists a constant $k \in (0,1)$ such that $F_{x_n,x_{n+1}}(kt) \ge F_{x_{n-1},x_n}(t)$ for all $t > 0$ and $n = 1,2\ldots$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.13[15]:- Let $(X,F,*)$ be a Menger space. If there exists $k \in (0,1)$ such that $F_{x,y}(kt) \ge F_{x,y}(t)$ for all $x,y \in X$ and $t > 0$, then $x = y$.

III. MAIN RESULTS

Theorem 3.1 Let A,B,S,T and P be self maps on a complete Menger space $(X,F,*)$ with $t * t \ge t$ for all $t \in [0,1]$, satisfying:

(a) P(X) ⊆ AB(X), P(X) ⊆ ST(X);

(b) there exists a constant k ∈ (0, 1) such that

$$M_{Px,Py,}(kt) \ge M_{ABx,Px,}(t) * M_{Px,STy,}(t) * M_{ABx,STy,}(t) * \frac{M_{Px,ABx,}(t)*M_{Px,STy,}(t)}{M_{STy,ABx,}(t)} * M_{ABx,Py,}(3-\alpha)t$$

$for\ all\ x,y \in X, \alpha \in (0,3)\ and\ t > 0,$

(c) PB = BP, PT = TP, AB = BA and ST = TS,

(d) A and B are continuous,

(e) the pair (P, AB) is compatible (if compatible then it is weak compatible)

Then A, B, S, T and P have a common fixed point in X.

Proof:- Since $P(X) \subset AB(X)$, for $x_0 \epsilon X, we\ can\ choose\ a\ point\ x_0 \epsilon X\ such\ that\ Px_0 = ABx_1. Since$ $P(X) \subset ST(X)$, for this point x_1, we can choose a point $x_2 \epsilon X$ such that

$Px_1 = STx_2$. Thus by induction, we can define a sequence $y_n \epsilon X$ as follows:

$$y_{2n} = Px_{2n} = ABx_{2n+1}$$

and

$$y_{2n+1} = Px_{2n+1} = STx_{2n+1}$$

for $n = 1,2, \ldots. By\ (b),$

For all $t > 0\ and\ \alpha = 2-q\ with\ q\epsilon(0,2), we\ have$

$$M_{y_{2n+1},y_{2n+2,}}(kt) = M_{Px_{2n+1},Px_{2n+2,}}(kt)$$

$$\ge M_{y_{2n+1},y_{2n+1,}}(t) * M_{y_{2n},y_{2n+1,}}(t) * M_{y_{2n},y_{2n+1,}}(t) * \frac{M_{y_{2n+1},y_{2n,}}(t)*M_{y_{2n+1},y_{2n+1,}}(t)}{M_{y_{2n+1},y_{2n,}}(t)} * M_{y_{2n},y_{2n+2,}}(1+q)t,$$

$$M_{y_{2n+1},y_{2n+2,}}(kt) \ge M_{y_{2n},y_{2n+1,}}(t) * M_{y_{2n},y_{2n+2,}}(1+q)t$$

$$\ge M_{y_{2n},y_{2n+1,}}(t) * M_{y_{2n},y_{2n+1,}}(t) * M_{y_{2n+1},y_{2n+2,}}(qt)$$

$$\ge M_{y_{2n},y_{2n+1,}}(t) * M_{y_{2n+1},y_{2n+2,}}(t)$$

as $q \to 1$.Since$*$is continuous and $M_{x,y}(*)$ is continuous, letting $q \to 1$ in above eq.,we get

$$M_{y_{2n+1},y_{2n+2,}}(kt) \ge M_{y_{2n},y_{2n+1,}}(t) * M_{y_{2n+1},y_{2n+2,}}(t) \ldots\ldots \quad (1)$$

Similarly, we have

$M_{y_{2n+2},y_{2n+3},}(kt) \geq M_{y_{2n+1},y_{2n+2},}(t) * M_{y_{2n+2},y_{2n+2},}(t)$ …… (2)

Thus from (1) and (2), it follows that

$M_{y_{n+1},y_{n+2},}(kt) \geq M_{y_n,y_{n+1},}(t) * M_{y_{n+1},y_{n+2},}(t)$

for $n = 1,2, \ldots$

and then for positive integers n and p,

$M_{y_{n+1},y_{n+2},}(kt) \geq M_{y_n,y_{n+1},}(t) * M_{y_{n+1},y_{n+2},}\left(\frac{t}{k^p}\right).$

Thus, since

$M_{y_{n+1},y_{n+1},}\left(\frac{t}{k^p}\right) \to 1 \; as \;\; p \to \infty$

we have

$M_{y_{n+1},y_{n+2},}(kt) \geq M_{y_n,y_{n+1},}(t).$

$\{y_n\}$ is Cauchy sequence in X and since x is complete, y_n converges to a point $z \in X$. Since Px_n, ABx_{2n+1} and STx_{2n+2} are subsequences of y_n, they also converge to the point z. Since A, B are continuous and pair $\{P, AB\}$ is compatible and also weak compatible, we have

$lim_{n\to\infty} \, PABx_{2n+1} = ABz$

and $lim_{n\to\infty}(AB)^2 \, x_{2n+1} = ABz.$

By (b) with $\alpha = 2$, we get

$M_{PABx_{2n+1}, \; Px_{2n+2},}(kt) \geq M_{(AB)^2x_{2n+1},}(t) * M_{PABx_{2n+1}, \; STx_{2n+2},}(t) * M_{(AB)^2x_{2n+1}, \; STx_{2n+2},}(t) * \frac{M_{PABx_{2n+1},(AB)^2 \, x_{2n+1},}(t) * M_{PABx_{2n+1}, \; STx_{2n+2},}(t)}{M_{STx_{2n+2},(AB)^2 \, x_{2n+1},}(t)} * M_{(AB)^2x_{2n+1}, \; Px_{2n+2},}(t)$

which implies that

$M_{ABz,z}(kt) = lim_{n\to\infty} \, M_{PABx_{2n+2},}(kt) \geq 1 * M_{ABz,z,}(t) * M_{ABz,z,}(t) * \frac{1 * M_{ABz,z}(t)}{M_{z,ABz}(t)} * M_{ABz,z,z}(t)$

we have $ABz = z$,since $M_{z,,STz}(t) \geq M_{z,ABz}(t) = 1$ for all $t > 0$, we get $STz = z$. Again by (b) with $\alpha = 2$, we have

$M_{PABx_{2n+1},Pz}(kt) \geq M_{(AB)^2x_{2n+1}, \; PABx_{2n+1},}(t) * M_{PABx_{2n+1}, \; STz,,}(t) * M_{(AB)^2x_{2n+1}, \; STz,,}(t) * \frac{M_{PABx_{2n+1},(AB)^2 \, x_{2n+1},}(t) * M_{PABx_{2n+1}, \; STz,,}(t)}{M_{STz,(AB)^2 \, x_{2n+1},}(t)} * M_{(AB)^2x_{2n+1}, \; Pz,}(t)$

which implies that

$M_{ABz,Pz,Pz}(kt) = lim_{n\to\infty} \, M_{PABx_{2n+1}, \, Pz,}(kt) \geq 1 * 1 * 1 * 1 * M_{ABz,Pz,}(t) \geq M_{ABz,Pz,}(t).$

we have $ABz = Pz$.Now, we show that $Bz = z$. Infact, by (b) with $\alpha = 2$, and (c) we get,$M_{Bz,z,}(kt) = M_{BPz,Pz,}(kt) = M_{PBz,Pz,}(kt)$

$M_{PBz,Pz,}(kt) \geq M_{PBz,STz,}(t) * M_{ABBz,STz,}(t) * \frac{M_{PBz,ABBz,}(t) * M_{PBz,z,z}(t)}{M_{z,PBz,}(t)} * M_{PBz,z}(t)$

$= 1 * M_{Bz,z,}(t) * M_{Bz,z,}(t) * 1 * M_{Bz,z,}(t) = M_{Bz,z,}(t).$

which implies that $Bz = z$.Since $ABz = z$, we have $Az = z$.Next, we show that $Tz = z$. Indeed by (b) with $\alpha = 2$, and (c) we get

$M_{Tz,z,}(kt) = M_{TPz,Pz,}(kt) = M_{Pz,Pz,}(kt)$

$\geq 1 * M_{z,Tz,}(t) * M_{z,Tz,}(t) * 1 * M_{z,Tz,}(t)$

$\geq M_{Tz,z,}(t)$,

which implies that $Tz = z$.Since $STz = z$, we have $Sz = STz = z$.Therefore, by combining the above results we obtain,

$Az = Bz = Sz = Tz = Pz$, that is z *is the* common fixed point of A, B, S, T and P.

Finally, the uniqueness of the fixed point *of* A, B, S, T and P.

COROLLARY:- Let $(X, F,*)$ be a complete Menger Space with $t * t \geq t$ for *all* $t \in [0,1]$, and let A, S and P be mappings from X into itself such that

(a) P(X) ⊆ A(X) and P(X) ⊆ S(X)

(b) there exists a constant $k \in (0,1)$ such that

$M_{Px,Py,}(kt) \geq M_{Ax,Px,}(t) * M_{Px,Sy,}(t) * M_{Ax,Sy,}(t) * \frac{M_{Px,Ax,}(t) * M_{Px,Sy,}(t)}{M_{Sy,Ax,}(t)} * M_{Ax,Py,}(3 - \alpha)t$

for all $x, y \in X, \alpha \in (0,3)$ *and* $t > 0$,

(c) A or P are continuous,

(d) the pair $\{P, A\}$ is compatible ,

Then A, S and P have a common fixed point in X.

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