ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $2x^2 - 3xy + 2y^2 = 56z^2$

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Abstract: The Homogeneous Ternary Quadratic Diophantine Equation is given by $2x^2 - 3xy + 2y^2 = 56z^2$ and analyzed for its patterns of non-zero distinct integer solutions. A few interesting relations among the solutions and special polygonal and pyramided numbers are presented. Introducing the linear transformation x=u+v,y=u-v and employing the method of factorization, different patterns of non-zero distinct integer solutions to the above equation are obtained.

Keywords: Homogeneous Quadratic, Ternary Quadratic, Integer solutions, polygonal number and pyramidal number

1. INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research by reason of their variety [1, 2]. In particular, one may refer [3, 19] for finding integer points on the some specific three dimensional surface. This communication concerns with yet another ternary quadratic Diophantine equation $2x^2 - 3xy + 2y^2 = 56z^2$ representing cone for determining its infinitely many integer solutions.

1.1 Notations Used:

1. Polygonal number of rank 'n' with sides' m

$$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$$

2. Stella Octangular number of rank 'n'

$$SO_n = n(2n^2 - 1)$$

3. Pyramidal number of rank 'n' sides' m

$$P_n^m = \frac{n(n+1)}{6}[(n-1)n + (5-m)]$$

4. Pronic number of rank 'n'

$$Pr_n = n(n+1)$$

2. METHOD OF ANALYSIS

Consider the equation $2x^2 - 3xy + 2y^2 = 56z^2$ (1)

The transformed equation of (1) after using the linear transformations $x = u + v$, $y = u - v$ (2)

$(u \neq v \neq 0)$ is $u^2 + 7v^2 = 56z^2$ (3)

The above equation is solved through different methods and using (2), different patterns of integer solutions to (1) are obtained.

2.1 PATTERN

Write 56 as $56 = (7 + i\sqrt{7})(7 - i\sqrt{7})$ (4)

Assume that $z = a^2 + 7b^2$ where a, b>0 (5)

Using (4) and (5) in (3) and applying the method of factorization, define

$$(u + i\sqrt{7}v) = (7 + i\sqrt{7})(a + i\sqrt{7}b)^2 \quad (6)$$

Equating the real and imaginary parts, we have

$u = u(a,b) = 7a^2 - 49b^2 - 14ab$

$v = v(a,b) = a^2 - 7b^2 + 14ab$

Substituting the above u and v in (2), the values of x and y are given by

$x = x(a,b) = 8a^2 - 56b^2$

$y = y(a,b) = 6a^2 - 42b^2 - 28ab$ (7)

Thus (5) and (7) represent non-zero distinct integral solution of equation (1) in two parameters.

PROPERTIES:

(1) $x(a,a) - y(a,a) - 16t_{4,a} = 0$

(2) $8z(a,1) - Pr_7 - 8t_{4,a} = 0$

(3) $z(a,a)-8t_{4,a}=0$

(4) $z(a,b)-t_{4,a}-7t_{4,b}=0$

(5) $x(a,a)+y(a,a)+112t_{4,a}=0$

2.2 PATTERN

Treating (1) as a Quadratic equation in x and solving for x, we get

$$x=\frac{1}{4}\left[3y\pm\sqrt{448z^2-7y^2}\right] \quad (8)$$

Let $\alpha^2=448z^2-7y^2$

$$=7(8z+y)(8z-y) \quad (9)$$

Write (9) in the form of ratio as

$$\frac{8z+y}{\alpha}=\frac{\alpha}{7(8z-y)}=\frac{A}{B}$$

This is equivalent to the following system of equations

$$A\alpha-yB-8zB=0$$

$$B\alpha+7Ay-56Az=0$$

On employing the method of cross multiplication, we get

$\alpha=112AB$

$y=56A^2-8B^2$

$z=7A^2+B^2$ (10)

Substituting the values of y and z from (10) in (8), the non-zero distinct integer values of x are given by $x=42A^2-6B^2\pm28AB$ (11)

Thus (10) and (11) represent the non-zero distinct integer solutions of equation (1) in two parameters

SET-1 $x=42A^2-6B^2+28AB$, $y=56A^2-8B^2$ and $z=7A^2+B^2$

PROPERTIES:

(1) $x(A,A)+y(A,A)-112t_{4,A}=0$

(2) $z(A,B)-7t_{4,A}-t_{4,B}=0$ (Works for the solution in set-2 also)

(3) $x(A,A)-y(A,A)\equiv 0 mod(16)$

(4) $y(A,A)-z(A,A)-40t_{4,A}=0$ (Works for the solution in set-2 also)

(5) $y(A,A)+z(A,A)-56t_{4,A}=0$ (Works for the solution in set-2 also)

SET-2 $x=42A^2-6B^2-28ABy=56A^2-8B^2$, and $z=7A^2+B^2$

PROPERTIES:

(1) $x(A,A)+y(A,A)-56t_{4,A}=0$

(2) $x(A,A)-y(A,A)+40t_{4,A}=0$

(3) $x(1,A)+y(1,A)+z(1,A)-64t_{4,A}=0$

(4) $y(1,A)-Pr_7+8t_{4,A}=0$

2.3 PATTERN

Equation (9) can also be expressed in the form of ratio as

$$\frac{7(8z-y)}{\alpha}=\frac{\alpha}{(8z+y)}=\frac{A}{B}$$

This is equivalent to the following system of equations

$$\alpha A+7B\mathbf{y}-56zB=0$$

$$B\alpha-Ay-8Az=0$$

On employing the method of cross multiplication, we get

$\alpha=112AB$

$y=56B^2-8A^2$

$z=A^2+7B^2$ (p_1)

Substituting the values of y and z from the above equations in (8), we get the non-zero distinct integer values of x are given by

$x=42B^2-6A^2\pm28AB$ (q_1)

Thus (p_1) and (q_1) represent the non-zero distinct integer solutions of equation (1) in two parameters

SET-1: $x=42B^2-6A^2+28AB$,$y=56B^2-8A^2$ and $z=A^2+7B^2$

SET-2 $x=42B^2-6A^2-28AB$, $y=56B^2-8A^2$ and $z=A^2+7B^2$

2.4 PATTERN

Equation (9) can also be expressed in the form of ratio as

$$\frac{7(8z+y)}{\alpha}=\frac{\alpha}{(8z-y)}=\frac{A}{B}$$

This is equivalent to the following system of equations

$$-\alpha A+7By+56zB=0$$

$$B\alpha+Ay-8Az=0$$

On employing the method of cross multiplication, we get

$\alpha=112AB$

$y=8A^2-56B^2$

$z=A^2+7B^2$ (p_2)

Substituting the values of y and z from the above equations in (8), we get the non-zero distinct integer values of x are given by

$x=6A^2-42B^2\pm 28AB$ (q_2)

Thus (p_2) and (q_2) represent the non-zero distinct integer solutions of equation (1) in two parameters.

SET-1 $x=6A^2-42B^2+28AB$, $y=8A^2-56B^2$ and $z=A^2+7B^2$

SET-2 $x=6A^2-42B^2-28AB$, $y=8A^2-56B^2$ and $z=A^2+7B^2$

2.5 PATTERN

Equation (9) can also be expressed in the form of ratio as

$$\frac{(8z-y)}{\alpha}=\frac{\alpha}{7(8z+y)}=\frac{A}{B}$$

This is equivalent to the following system of equations

$$-\alpha A-By+8zB=0$$

$$-B\alpha+7Ay+56Az=0$$

On employing the method of cross multiplication, we get

$\alpha=112AB$

$y=8B^2-56A^2$

$z=7A^2+B^2$ (p_3)

Substituting the values of y and z from the above equations in (8), we get the non-zero distinct integer values of x are given by

$x=6B^2-42A^2\pm 28AB$ (q_3)

Thus (p_3) and (q_3) represent the non-zero distinct integer solutions of equation (1) in two parameters.

SET-1 $x=6B^2-42A^2+28AB$, $y=8B^2-56A^2$ and $z=7A^2+B^2$

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2.6 PATTERN

Rewrite equation (3) as $7v^2=56z^2-u^2$ (12)

Write $7=\left(2\sqrt{14}+7\right)\left(2\sqrt{14}-7\right)$ (13)

Let $v=56a^2-b^2$, where a, b>0 (14)

Using (13) and (14) in (12) and employing the method of factorization, we write

$$2\sqrt{14}z+u=\left(2\sqrt{14}+7\right)\left(2a\sqrt{14}+b\right)^2$$

Equating the rational and irrational parts on both sides, we have

$z=z(a,b)=56a^2+b^2+14ab$ (15)

$u=u(a,b)=392a^2+7b^2+112ab$ (16)

Thus substituting (14) and (16) in (2), the value of x and y are

$$x=x(a,b)=448a^2+6b^2+112ab$$

$y=y(a,b)=336a^2+8b^2+112ab$ (17)

PROPERTIES:

(1) $x(a,a)+y(a,a)+2t_{4,a}\equiv 0 mod(4)$

(2) $z(a,a)-7t_{4,a}$ *is a perfect square*.

(3) $x(1,a)+y(1,a)\equiv 0 mod(14)$

(4) $z(1,a)-Pr_7-7t_{4,a}-SO_2=0$

(5) $x(1,a)+y(1,a)-14z(1,a)\equiv 0 mod(28)$

3. CONCLUSION

In this paper, I have presented different pattern of integer solutions to the ternary quadratic Diophantine equation $2x^2-3xy+2y^2=56z^2$ representing the cone. As these Diophantine equations are rich in variety, one may attempt to find integer solutions to other choices of equations along with suitable properties.

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AUTO-BIOGRAPHY

Chikkavarapu Gnanendra Rao,(born on 31-1-83 in Nuzvid, Andhra Pradesh) completed post-graduation in mathematics from P.B.Siddartha College of arts and science in Vijayawada in the year 2005. Presently he is working as assistant professor of mathematics in Sri Sarathi Institute of Engineering and Technology, Nuzvid. He was qualified in GATE 2013 with 93.71 percentile and APSET-2012.