# MATHEMATICAL MODELLING OF GROUNDWATER FLOW
# A Case Study of Flow of Water under a Dam

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**Abstract -** *The study considered two dimensional steady state groundwater flows. The general objective of this study was to develop and analyze a mathematical model of groundwater flow under a dam using the finite element method. Simulation of the model was carried out to determine the relationship between the hydraulic pressure and groundwater flow in aquifer using the finite element method. The study focused on groundwater flow under a dam. The result indicates that groundwater flow under a dam depends on the distribution of hydraulic pressure in the dam. Moreover, the study shows that high flow velocity of groundwater around the weirs can cause dam failure if the water level is not controlled*

***Keywords:*** **Dam, Finite Element**

## 1.0 Introduction

The movement of groundwater through the soil and bedrocks is termed as groundwater flow. The particles and bedrocks which retain groundwater are called aquifers [1]. Groundwater is considered potentially good and useful for domestic purposes or used to augment surface water supplies in water schemes, irrigation, industrial and livestock uses [2]. In many countries, groundwater is the foundation on which agriculture; urban development, rural jobs, and safe drinking water supply systems have been built [3].

Current stresses on surface water resources are causing a serious depletion of groundwater. Despite many activities involves groundwater extraction, Human being construct dams to obtain water as a solution to the problem of scarcity of surface water. The flow of water under a dam is therefore important groundwater flow problems that are governed by the groundwater flow equation, and which need to be studied and analyzed.

## 2.0 Flow of Water under a Dam

Dams consist of two main parts, the reservoir or impoundment and the outer that are separated by a dam wall which is usually a concrete, rock or earth filled barrier placed across a river to control the passage of water. The lower structure of the dam wall is known as weirs, and it may be only a few meters in height. The reservoir or impoundment is the water body retained behind the dam.

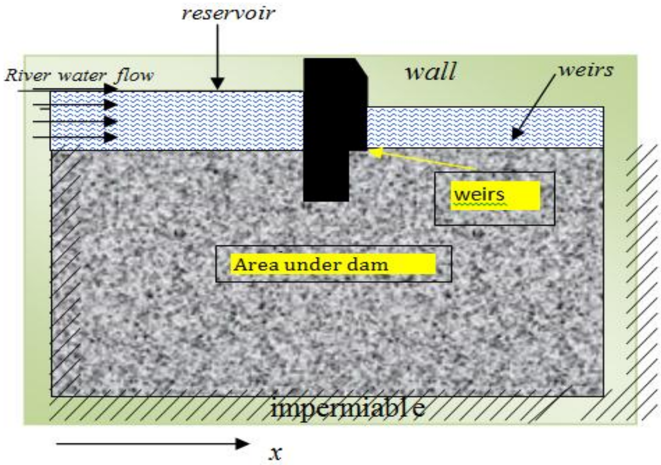


Figure1. 1: Main Features of a Dam

[4] stated that groundwater flow under a dam is represented by the continuity equation which is the fundamental basis upon which the concept of drawing flow nets is derived. Flow nets are powerful tools for calculation of seepage as well as the uplift pressure under various hydraulic structures. [4] further discovered that a dam causes deepening of a riverbed, and this in turn lowers groundwater tables along the river, hence making the water table inaccessible to plant roots (and to human communities drawing water from wells). Altering the riverbed also reduces the habitat area for fish that spawn in river bottoms, and for invertebrates.

Despite the constructions of dams, the quantity of water in dams usually decreases due to the seepage process that gives rise to shortage of water. Dam constructions activities are being done while people have poor knowledge of groundwater flow. [5] reported that all the on-going groundwater resources development in Tanzania is being carried out without sufficient knowledge of the resource potential. This study aims to model groundwater flow under a dam using the finite element method.

## 3.0 Governing Equations for Groundwater Flow under a dam

The standard equations that govern groundwater flow are derived using the principle of continuity and Darcy's law. Regardless of the type of dam, the seepage of ground water under dams for the two dimensions in a steady state condition is governed by the Laplace equation [6]

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} = 0 \qquad (3.1)$$

where $K_x$ and $K_y$ are the coefficients of permeability (m/day) and h is the piezo - metric head in meters, measured from the bottom of the confined aquifer.

## 3.1 Formulation of the Governing Differential Equation

The solution $h(x, y)$ of the problem described by equation (3.1) will be sought in a function space $P$ known as the trial space using standard Galerkin finite element method. If $h$ and V are any two functions in $P$ one forms the inner product [6]

$$\iint_{\Omega} \left( K_X \frac{\partial^2 h(x,y)}{\partial x^2} + K_y \frac{\partial^2 h(x,y)}{\partial y^2} \right) V(x,y)\,dxdy = 0 \qquad (3.2)$$

Applying Green's theorem to the integral we obtains the corresponding Galerkin weak formulation. Find the unique function $h(x, y) \in P$ such that $B(h, V) = 0, \quad \forall_v \in P$, where $B(h, V)$ is a bilinear form defined by

$$B(h,V) = \int_{\partial\Omega} \left( K_X \frac{\partial h}{\partial x} + K_y \frac{\partial h}{\partial y} \right) V(x,y)\,dS - \iint_{\Omega} \left( K_X \frac{\partial h}{\partial x}\frac{\partial V}{\partial x} + K_y \frac{\partial h}{\partial y}\frac{\partial V}{\partial y} \right) dxdy$$

(3.3)

We can write equation (3) as

$$\iint_{\Omega} \left( K_x \frac{\partial h}{\partial x}\frac{\partial V}{\partial x} + K_y \frac{\partial h}{\partial y}\frac{\partial V}{\partial y} \right) dxdy = \int_{\partial\Omega} \left( K_X \frac{\partial h}{\partial x} + K_y \frac{\partial h}{\partial y} \right) V(x,y)\,dS$$

(3.4)

Since the integral is a bilinear function, linear in both h and V and symmetrical, we make the substitution

$$h = \sum_{n=1}^{N} h_n Q_n \quad \text{into (4)}$$

$$B(h_1\varphi_1(x,y) + h_2\varphi_2(x,y) + h_3\varphi_3(x,y) + \ldots + h_n\varphi_n(x,y)), V) = \int_{\partial\Omega} (K_x \bar{q}_1 + K_y \bar{q}_2)\varphi_n\, dS$$

(3. 5)

The system of equations (3.5) represents the finite element mathematical model of the groundwater, simplified in system matrix equation as [3.6]

M h = f (3.6)

where, M is the global stiffness matrix

h is a column vector whose elements are the nodal solution

f is the vector in which the boundary conditions are incorporated.

### 3.1.1. The Global Stiffness Matrix

The element stiffness matrix for domain under a dam is evaluated as

$$Me = \frac{1}{2}\begin{bmatrix} K_X + K_Y & -K_X & -K_Y \\ -K_X & K_X & 0 \\ -K_Y & 0 & K_Y \end{bmatrix} \qquad (3.7)$$

Element stiffness matrix contributes three rows and three columns to the global stiffness matrix. Then, global stiffness matrix for the domain under a dam is given by

$$\begin{array}{cc} & \begin{array}{ccccccccc} 1 & \cdots & i & \cdots & j & \cdots & k & \cdots & N \end{array} \\ M = \begin{array}{c} 1 \\ \vdots \\ i \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \\ N \end{array} & \frac{1}{2}\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & K_x + K_y & \cdots & -K_x & \cdots & -K_y & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -K_X & \cdots & K_x & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -K_y & \cdots & 0 & \cdots & K_y & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

(3.8)

$$M = \sum_{e=1}^{e} M^{e}_{n,i}$$

For all n and i, note that $M_{n,i} = M_{i,n}$ is a symmetric matrix.

### 3.1.2. Incorporating the Boundary Conditions

The flow boundary conditions are incorporated into the column vector **f** of the system of equation (3.6). For all interior

nodes or nodes on a no- flow boundary, $f_n$ = 0. The boundary nodes on a specified head, $f_n$ is evaluated from the integral given on right hand side of equation (3.8). The condensed force vector with the positions of its term in the global force can be written in matrix form as [7]

$$\mathbf{f} = \begin{bmatrix} \int_{\partial\Omega} \bar{q}\varphi_1 dS \\ \int_{\partial\Omega} \bar{q}\varphi_2 dS \\ \vdots \\ \int_{\partial\Omega} \bar{q}\varphi_n dS \end{bmatrix} \qquad (3.9)$$

### 3.1.3. Solving the System of Equations for the Unknown Solution

Using MATLAB codes the system of the equation (3.6) assembled from the global matrices equation (3.8) and the condensed force matrix (3.9). Furthermore the MATLAB codes will be used to define the unknown hydraulic heads in column vector h and provide graphical solution of groundwater flow under a dam.

## 4.0 Numerical Simulations of the Model and Discussion

### 4.1 Material Properties of an Aquifer

Groundwater flow under a dam in a steady-state condition is the Laplace equation. The study considered the case of two dimensional groundwater flow in an aquifer where the material properties (permeability), hydraulic heads and fluxes of the area under a dam, were approximated.

### 4.2 Modelling Groundwater Flow under a Dam

The problem of water seepage under a dam where water flows to a reservoir or impoundment in a dam at a vertical height of 10.00 m, and the water level upon exiting the dam is at 2.00 m considered. The bottom of the dam rests on impermeable bedrock and the vertical boundary is assumed to be no flow. The permeability in the x and y directions is approximated at the rate of 200 $m^3$/day.

To investigate the patterned of the groundwater flow under a dam, equation (3.6) were used to determine the hydraulic heads at each point in the domain. Graphical solutions on the domain were obtained using MATLAB program. Figure 4.1 indicates equipotential lines representing points of equal head in aquifer while the arrows show the flow direction velocity for the water flow.

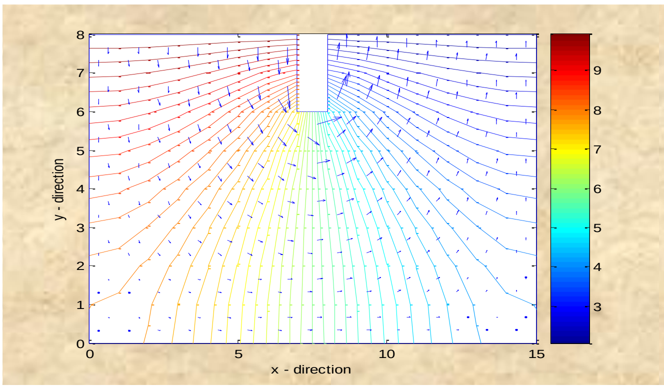


Figure 4.1: Equipotential lines and streamlines under a dam

The distribution of hydraulic pressure under the dam indicated in the Figure 4.1 shows that the equipotential lines decrease gently from the reservoir at vertical height of 10.00 m to outer with 2.00 m water level. Also, the streamlines lines (blue arrows) show the path followed by the water particles as it moves through the aquifer with decreasing head. Also, it is observed that the stream lines are higher around the weirs means there is higher velocity around the weirs that can cause the failure of the dam if the piezometric head of the dam not controlled.

### 4.3 Sensitivity Test of the Model

In this section we study the dependence of groundwater flow on the boundary conditions prescribed at the boundaries of the domain under a dam. In particular we investigate how small variations in permeability affect the hydraulic pressure of groundwater.

### 4.3.1. Effect of Varying the Permeability of an Aquifer on the Groundwater Flow

Small variations of permeability either in the x or in the y direction in an aquifer under a dam were made to study their impact on the flow of groundwater flow. All other parameters (piezometric head and boundary conditions) were kept fixed. Graphical results from obtained using the MATLAB codes are shown in the Figures 4.2, 4.3 and 4.4.

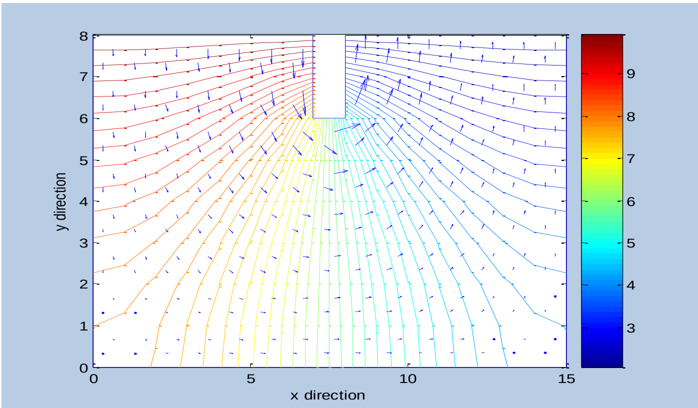


Figure 4.2: Equipotential lines and streamlines for $K_x = K_y = 200$ $m^3$/day.

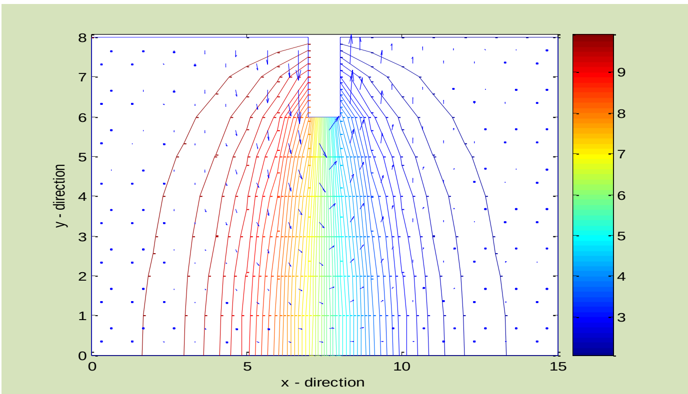


Figure 4.3: Equipotential lines and stream lines for $K_x \leq K_y$

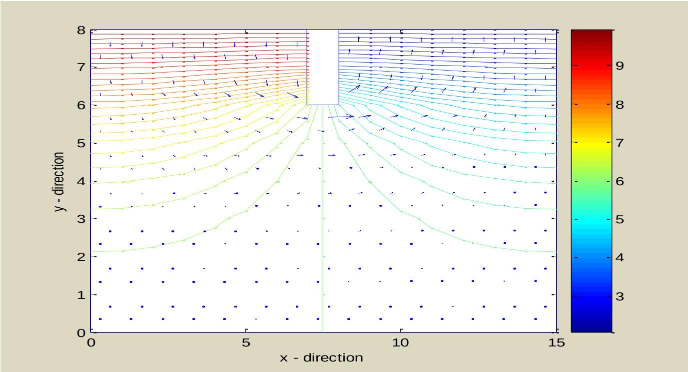


Figure 4.4: Equipotential lines and stream lines for $K_x \geq K_y$

These results show that groundwater flow in the aquifer is sensitive to small changes in the permeability parameters. Small change in permeability cause large changes in the direction of groundwater flow. Figure 4.2 shows that the flow of groundwater is towards the x and y directions, this is due to the equal permeability while Figure 4.3 shows that the flow is in the y direction due to its higher permeability, and Figure 4.4 shows that the flow is in the x direction due to higher permeability in this direction. These results imply that permeability of an aquifer influences the direction of ground water flow.

## 5.0 Conclusion

Groundwater flow under a dam is governed by the Laplace equation with Dirichlet boundary conditions. Using the finite element method the hydraulic heads distribution was found in the domain. The results show that groundwater flow under a dam depends on the hydraulic pressure in the dam, and this implies that the hydraulic pressure is the main cause of groundwater flow under a dam.

Sensitivity test of the model was carried out by making small changes in the boundary values. The results given in Figures 4.1, 4.2, 4.3 and 4.4 show that as the boundary values of a dam are increased the hydraulic pressure distributions in the dam also increases and vice versa. It is also observed that higher pressure exerted around the weirs may cause failure of the dam if the level of the dam is not controlled.

## 6.0 Recommendations

The results obtained in this study constitute an important first step in the study of groundwater flow in Tanzania. It is an important area of interest because it is concerned with the study of supply of water for domestic industrial and agricultural purposes.

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