ON SEMIGROUP AND ITS CONNECTIONS WITH LATTICES

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ABSTRACT:- *L. V. Shivrin and B. M. Vernikov derived properties of semi group varieties which forms a lattice. We critically examine the different classifications of modular varieties. It is known that the collection SEM of all semigroup varieties forms a lattice with respect to class theoretical inclusion. A semigroup variety modular is called lower-modular, distributive if it is a modular lower-modular, distributive element of the lattice SEM. Distributive varieties have been determined L.N. Shervin we discuss properties of a class of lower-modular varieties. B. V. Vernikov derived its properties, we examine a complete classification of lower-modular varieties. The main result of this article gives a complete classification of lower-modular varieties. If B is a set of identities, then* $\Leftrightarrow_B$ *denotes the fully invariant congruence on the free semigroup corresponding to B. we establish the connections between the quasi-orders* $\leq_B$ *and* $\leq$.

KEYWORDS: Semigroup, Lattices, Semilattice, Modular, Quasi-orders

INTRODUCTION

An element x of a lattice $\langle L, \vee, \wedge\rangle$ is called modular if

$\forall y,z \in L : y \leq z \rightarrow (x \vee y) \wedge z = (x \wedge z) \vee y,$ …(i)

lower-modular if

$\forall y,z \in L : x \leq y \rightarrow x \vee (y \wedge z) = y \wedge (x \vee z),$ …(ii)

distributive if

$\forall y,z \in L : x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$ …(iii)

Upper-modular elements may be defined dually to lower-modular ones. Hence, we find that a distributive element is lower-modular. A pair of identities wx = xw = w is known as an identity w = 0. Since justified because a semigroup with such identities has a zero element and all values of the word w in this semigroup are equal to zero. Identities of the form w = 0 moreover as varieties given by such identities are called 0-reduced. By T, SL, and SEM we denote the trivial variety, the variety of all semilattices, and the variety of all semigroups, respectively. We prove here the following theorem. Let us represent by F_∞ the free semigroup above a countably infinite alphabet, i.e. the semigroup of words under concatenation. If B is a set of regular identities and v and u are words, let us define $v \leq_B u$ if and only if var{$v \approx 0$,B} $|= u \approx 0$. If the set B contains only trivial identities, then instead of ≤B Let us write ≤. The relation '≤' on the free semigroup is known and may be defined as follows: if $u, v \in F_\infty$, then $v \leq u$ if and only if $u = a\Theta(v)b$ for some possibly empty words a and b and some substitution Θ we find that the relation '≤' B is reflexive and transitive, i.e. it is a quasiorder on the free semigroup F_∞. If $u \leq_B v \leq_B u$, then $u \Leftrightarrow_B v$. If u is a word, then the class of all words equivalent to u modulo $\Leftrightarrow_B$ is denoted by $[u]\Leftrightarrow_B$. Let $F_\infty/\Leftrightarrow_B$ denote the set of all classes $[u]\Leftrightarrow_B$ ordered by $\leq_B$. the elements of the ordered set $F_\infty/\Leftrightarrow_B$ is called *word patterns modulo* B.

Theorem 1

If $u \leq v$, *then* $|u| - |\mathrm{Cont}(u)| \leq |v| - |\mathrm{Cont}(v)|$.

Proof

Assertion of the lemmas is verified by taking the following two cases.

Case i: $v = ux$ or $v = xu$ for some new variable x.
Case ii: $v = \Theta(u)$ where Θ is an elementary substitution.

Also, the general case follows by transitivity of relation '≤' on word pattern.

Theorem 2

If B *is a* set of regular identities, then the following statements are equivalent:

(i) $v \leq_B u$;
(ii) *there exists a word u′ such that* $u \sim B\ u'$ *and* $v \leq u'$.

We first prove implication (i) ⟹ (ii).

Proof

(i) → (ii), If $v \leq B\ u$, then there exists a sequence of words $u_1, \ldots, u_n$ such that $u \approx 0$ from $\{v \approx 0, B\}$ which it follows that $u = u_1 \approx u_2 \approx \cdots \approx u_n \approx 0$. Since $m(B) > 0$ but $m(u \approx 0) = 0$, $u \approx 0$ is not derivable from B without using $v \approx 0$. By using the identity $v \approx 0$ the derivation of $u \approx 0$ is possible. So, $u \sim B\ u_n$ and $v \leq u_n$.
(ii)→(i). follows as an immediate consequence of condition (i). If $v \leq B\ u$ but u ⍰ B v, then we write $v < B\ u$.

Theorem 3

If B *is a set of balanced identities, then the following conditions are satisfied.*

(i) $u \Leftrightarrow_B v$ *if and only if* $u \sim_B p(v)$ *for some renaming p of variables*;
(ii) *if* $u \Leftrightarrow_B v$, *then* $|u| = |v|$;
(iii) *if* $v \leq_B u$ *and* $|v| < |u|$, *then* $v <_B u$;
(iv) $v <_B u$ *if and only if there exists a word u′ such that* $u \sim B\ u'$ *and* $v < u'$

Proof

We prove each statement separately.
(i) If $v \sim B\ p(u)$, then $u \approx p(v) \approx 0$ is a derivation of $u \approx 0$ from $\{v \approx 0, B\}$. The identity $v \approx 0$ can be derived from $\{u \approx 0, B\}$ in a similar way. So, $u \Leftrightarrow_B v$. If $v \Leftrightarrow B\ u$, then using assertion, let us obtain words u' and v' such that $u \sim_B u'$, $v \leq u'$ and $v \sim_B v'$, $u \leq v'$. Since the identities $u \approx u'$ and $v \approx v'$ are balanced, all the words u, u', v and v' have the same length. Therefore, $u' = p(v)$ for some substitution p that maps variables to variables. Let us suppose that for some distinct variables x and y with $\mathrm{occ}v(x) > 0$ and $\mathrm{occ}v(y) > 0$ we have that $p(x) = p(y) = z$. But then $|u| - |\mathrm{Cont}(u)| = |u'| - |\mathrm{Cont}(u')| > |v| - |\mathrm{Cont}(v)| = |v'| - |\mathrm{Cont}(v')|$. By an appropriate application of Lemma (4.3.5), this contradicts the fact that $u \leq v'$. Therefore, the substitution p is a renaming of variables.
(ii) If $u \Leftrightarrow_B v$, then by using part (i) the identity $u \approx p(v)$ is balanced. Therefore, $|u| = |p(v)| = |v|$.
(iii) If $u \Leftrightarrow_B v$, then by part (ii) we obtain that $|u| = |v|$. Therefore $v < B\ u$.
(iv) Since $v < B\ u$, By suitable application of lemma (4.3.5) we may find a word u' such that $u \sim B\ u'$ and $v \leq u'$. So, $u' = a\Theta(v)b$ for some possibly empty word ab and a substitution Θ. If the word ab is empty and Θ is a renaming of variables, then, by using part (i), we find that $u \Leftrightarrow_B v$. Since the words v and u are not equivalent modulo ⇔B, it implies $v < u'$.
Let us suppose that there is a word u' such that $u \sim B\ u'$ and $v < u'$. Then by an application of Lemma (4.3.5), we get $v \leq_B u$. If $|v| < |u|$ then by suitable application of part (iii), we find that $v <_B u$. Let us assume that $|u| = |v|$ and consequently $|u| = |v| = |u'|$. Since $v < u'$ and $|v| = |u'|$ we have that $u' = \Theta(v)$ for some substitution Θ that maps variables to variables such that Θ is not a renaming of variables. So, $u \sim_B \Theta(v)$. Let us further assume that $u \Leftrightarrow_B v$ then by part (i) that $u \sim_B p(v)$ for some

renaming of variables p. Thus the identity $\Theta(v) \approx p(v)$ also balanced which is not the case. Therefore, $v <_B u$. Hence, the lemma is established.

Theorem 4

If B *is a set of balanced identities with* ℓ(B) > 0, *then*:

(i) *the words* $x^2y_1 \cdots y_k$ *and* $y_1 \cdots y_kx^2$ *are not equivalent modulo* $\Leftrightarrow_B$ *for any* $k > 0$;
(ii) *the words* x^ky *and* yx^k *are not equivalent modulo* $\Leftrightarrow_B$ *for any* $k > 0$;
(iii) *the words* $y_1 \cdots y_j\, x^{k+2}t_1 \cdots t_p$ *and* $y_1 \cdots y_{j+1}x^{k+1}t_1 \cdots t_p$ *are not equivalent modulo* $\Leftrightarrow_B$ *for any* $k \geq 0$.

Proof

(i) If $x^2y_1 \cdots y_k \Leftrightarrow_B y_1 \cdots y_kx^2$, then Lemma (4.3.5) implies $x^2y_1 \cdots y_k \sim_B p(y_1 \cdots y^kx_2)$ for some renaming of variables p. Since x is the only non-linear variable in both words, this can only happen if B $\vdash x^2y_1 \cdots y_k \approx y_1 \cdots y^kx^2$ which is impossible in view of Lemma (4.3.5).
(ii) Similar to the proof of (i).
(iii) If $y_1 \cdots y_j\, x^{k+2}t_1 \cdots t_p \Leftrightarrow_B y_1 \cdots y_{j+1}x^{k+1}t_1 \cdots t_p$ then by Lemma (4.3.5) we have that $y_1 \cdots y_j\, x^{k+2}t_1 \cdots t_p \sim_B q(y_1 \cdots y^{j+1}x^{k+1}t_1 \cdots t_p)$ for some renaming of variables q. Since x is the only non-linear variable in both words and occurs different number of times in each of the word, this is impossible in view of Lemma (4.3.6).

Theorem 5

If L is a lattice and a ∈ L then [a) stands for the principal coideal generated by a, that is, the set {x ∈ L | x ≥ a}. If x is a lower-modular element of a lattice L and a ∈ L then the element x ≥ a is a lower-modular element of the lattice [a).

Proof

Let y, z ∈ [a) and x ∧ a ≤ y. Then, we have

$$\begin{aligned}(x \vee a) \vee (y \wedge z) &= a \vee \big(x \vee (y \wedge z)\big)\\ &= a \vee \big(y \wedge (x \vee z)\big)\\ &= y \wedge (x \vee z)\\ &= y \wedge \big(x \vee (a \vee z)\big)\\ &= y \wedge ((x \vee a) \vee z)\end{aligned}$$

Thus (x ∨ a) ∨ (y ∧ z) = y ∧ ((x ∨ a) ∨ z), Hence, the theorem is proved.

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