A generalized metric space and related fixed point theorems

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Abstract:

We introduce a new concept of generalized metric spaces for which we extend some well-known fixed point results including Banach contraction principle, Ćirić's fixed point theorem, a fixed point result due to Ran and Reurings, and a fixed point result due to Nieto and Rodríguez-López. This new concept of generalized metric spaces recover various topological spaces including standard metric spaces, *b*-metric spaces, dislocated metric spaces, and modular spaces.

Keywords:generalized metric, *b*-metric, dislocated metric, modular spacefixed point, partial order.

1. Introduction:

The concept of standard metric spaces is a fundamental tool in topology, functional analysis and nonlinear analysis. This structure has attracted a considerable attention from mathematicians because of the development of the fixed point theory in standard metric spaces.

In this work, we present a new generalization of metric spaces that recovers a large class of topological spaces including standard metric spaces, *b*-metric spaces, dislocated metric spaces, and modular spaces. In such spaces, we establish new versions of some known fixed point theorems in standard metric spaces including Banach contraction principle, Ćirić's fixed point theorem, a fixed point result due to Ran and Reurings, and a fixed point result due to Nieto and Rodíguez-López.

2. A generalized metric space:

Let X be a nonempty set and $D:X\times X\to[0,+\infty]$ be a given mapping. For every $x\in X$, let us define the set

$$C(D,X,x)=\{\{x_n\}\subset X:\lim_{n\to\infty}D(x_n,x)=0\}.$$

2.1 General definition

Definition 2.1

We say that D is a generalized metric on X if it satisfies the following conditions:

$(D_1)$for every $(x,y)\in X\times X$, we have

$$D(x,y)=0\Longrightarrow x=y;$$

$(D_2)$for every $(x,y)\in X\times X$, we have

$$D(x,y)=D(y,x);$$

(D_3)there exists $C>0$ such that

if $(x,y)\in X\times X,\{x_n\}\in C(D,X,x)$,then $D(x,y)\leq C\lim_{n\to\infty}\sup D(x_n,y)$.

In this case, we say the pair (X,D) is a generalized metric space.

Remark 2.2

Obviously, if the set $C(D,X,x)$ is empty for every $x\in X$, then (X,D) is a generalized metric space if and only if (D_1) and (D_2) are satisfied.

Topological concepts

Definition 2.3

Let (X,D) be a generalized metric space. Let $\{x_n\}$ be a sequence in *X* and $x \in X$. We say that $\{x_n\}$ D-converges to *x* if $\{x_n\} \in C(D,X,x)$.

Proposition 2.4

Let(X,D)*be a generalized metric space*. *Let*$\{x_n\}$ *be a sequence inXand*$(x,y) \in X \times X$. *If*$\{x_n\}$ D-*converges toxand*$\{x_n\}$ D-*converges toy*, *then*x=y.

Proof

Using the property (D_3), we have

$$D(x,y) \leq \lim_{n\to\infty} \sup D(x_n,y)=0,$$

which implies from the property (D_1) that x=y.

Definition 2.5

Let (X,D) be a generalized metric space. Let $\{x_n\}$ be a sequence in *X*. We say that $\{x_n\}$ is a D-Cauchy sequence if

$$\lim_{m,n\to\infty} D(x_n,x_{n+m})=0.$$

Definition 2.6

Let (X,D) be a generalized metric space. It is said to be D-complete if every Cauchy sequence in *X* is convergent to some element in *X*.

Standard metric spaces

Recall that a standard metric on a nonempty set *X* is a mapping $d:X\times X\to[0,+\infty)$satisfying the following conditions:

*(d_1)*for every $(x,y)\in X\times X$, we have
$d(x,y)=0 \Leftrightarrow x=y$;

*(d_2)*for every $(x,y)\in X\times X$, we have
$d(x,y)=d(y,x)$;

*(d_3)*for every $(x,y,z)\in X\times X\times X$, we have
$d(x,y)\leq d(x,z)+d(z,y)$.

It is not difficult to observe that *d* satisfies all the conditions (D_1)-(D_3) with C=1.

***b*-Metric spaces**

In 1993, Czerwik [1] introduced the concept of *b*-metric spaces by relaxing the triangle inequality as follows.

Definition 2.7

Let *X* be a nonempty set and $d:X\times X\to[0,+\infty)$ be a given mapping. We say that *d* is a *b*-metric on *X* if it satisfies the following conditions:

*(b_1)*for every $(x,y)\in X\times X$, we have

$$d(x,y)=0 \Leftrightarrow x=y;$$

*(b_2)*for every $(x,y)\in X\times X$, we have
$d(x,y)=d(y,x)$;

*(b_3)*there exists $s\geq 1$ such that, for every $(x,y,z)\in X\times X\times X$, we have
$d(x,y)\leq s[d(x,z)+d(z,y)]$.

In this case, (X,d) is said to be a *b*-metric space.

The concept of convergence in such spaces is similar to that of standard metric spaces.

Proposition 2.8

Anyb-metric onXis a generalized metric onX.

Proof

Let *d* be a *b*-metric on *X*. We have just to proof that *d* satisfies the property (D_3). Let $x\in X$ and $\{x_n\}\}\in C(d,X,x)$. For every $y\in X$, by the property (b_3), we have
$d(x,y)\leq sd(x,x_n)+sd(x_n,y)$,for every natural number *n*. Thus we have

$d(x,y)\leq s\lim_{n\to\infty}\sup d(x_n,y)$.

The property (D_3) is then satisfied with C=s.

Hitzler-Seda metric spaces

Hitzler and Seda [8] introduced the notion of dislocated metric spaces as follows.

Definition 2.9

Let *X* be a nonempty set and $d:X\times X\to[0,+\infty)$ be a given mapping. We say that *d* is a dislocated metric on *X* if it satisfies the following conditions:

*(HS_1)*for every $(x,y)\in X\times X$, we have

$$d(x,y)=0\Longrightarrow x=y;$$

*(HS_2)*for every $(x,y)\in X\times X$, we have

$$d(x,y)=d(y,x);$$

*(HS_3)*for every $(x,y,z)\in X\times X\times X$, we have

$$d(x,y)\leq d(x,z)+d(z,y).$$

In this case, (X,d) is said to be a dislocated metric space.

The motivation of defining this new notion is to get better results in logic programming semantics.

The concept of convergence in such spaces is similar to that of standard metric spaces.

The following result can easily be established, so we omit its proof.

Proposition 2.10

Any dislocated metric onXis a generalized metric onX.

Modular spaces with the Fatou property

Definition 2.11

Let *X* be a linear space over R. A functional $\rho:X\to[0,+\infty]$ is said to be modular if the following conditions hold:

*(ρ1)*for every $x\in X$, we have

$$\rho(x)=0\Longleftrightarrow x=0;$$

*(ρ2)*for every $x \in X$, we have

$\rho(-x)=\rho(x)$;

*(ρ3)*for every $(x,y) \in X \times X$, we have

$\rho(\alpha x+\beta y) \le \rho(x)+\rho(y)$,

whenever $\alpha, \beta \ge 0$, and $\alpha+\beta=1$.

Definition 2.12

If ρ is a modular on X, then the set

$$X_\rho=\{x \in X: \lim_{\lambda \to 0} \rho(\lambda x)=0\}$$

is called a modular space.

The concept of convergence in such spaces is defined as follows.

Definition 2.13

Let (X,ρ) be a modular space.

1. (i)A sequence $\{x_n\}_{n \in N} \subset X_\rho$ is said to be ρ-convergent to $x \in X_\rho$ if $\lim_{n\to\infty}\rho(x_n-x)=0$.

2. (ii)A sequence $\{x_n\}_{n \in N} \subset X_\rho$ is said to be ρ-Cauchy if $\lim_{n,m\to\infty}\rho(x_n-x_{n+m})=0$.

3. (iii)X_ρ is said to be ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

Definition 2.14

The modular ρ has the Fatou property if, for every $y \in X_\rho$, we have

$$\rho(x-y) \le \lim_{n\to\infty} \inf \rho(x_n-y),$$

whenever $\{x_n\}_{n \in N} \subset X_\rho$ is ρ-convergent to $x \in X_\rho$.

Let (X,ρ) be a modular space. Define the mapping $D_\rho: X_\rho \times X_\rho \to [0,+\infty]$ by
$D_\rho(x,y)=\rho(x-y)$,for every $(x,y) \in X \times X$.

We have the following result.

Proposition 2.15

Ifρhas the Fatou property, thenD_ρ*is a generalized metric on*X_ρ.

Proof

We have just to proof that D_ρ satisfies the property (D_3). Let $x \in X_\rho$ and $\{x_n\} \in C(D_\rho, X_\rho, x)$, which means that

$$\lim_{n\to\infty}\rho(x_n-x)=0.$$

Using the Fatou property, for all $y \in X_\rho$, we have
$\rho(x-y) \le \lim_{n\to\infty} \inf \rho(x_n-y)$,

which yields
$D_\rho(x,y) \le \lim_{n\to\infty} \inf D_\rho(x_n,y) \le \lim_{n\to\infty} \sup D_\rho(x_n,y)$.

Then (D_3) is satisfied with $C=1$ and D_ρ is a generalized metric on X_ρ.

The following result is immediate.

We have the following extension of the Banach contraction principle.

3. MAIN RESULT:

The Banach contraction principle in a generalized metric space:

In this section, we present an extension of the Banach contraction principle to the setting of generalized metric spaces introduced previously.

Let (X,D) be a generalized metric space and $f:X\to X$ be a mapping.

Definition 3.1

Let $k\in(0,1)$. We say that f is a k-contraction if

$D(f(x),f(y))\leq kD(x,y)$, for every $(x,y)\in X\times X$.

First, we have the following observation.

Proposition 3.2

Suppose that f is a k-contraction for some $k\in(0,1)$. *Then any fixed point* $\omega\in X$ *of f satisfies*

$D(\omega,\omega)<\infty\Longrightarrow D(\omega,\omega)=0$.

Proof

Let $\omega\in X$ be a fixed point of f such that $D(\omega,\omega)<\infty$. Since f is a k-contraction, we have

$D(\omega,\omega)=D(f(\omega),f(\omega))\leq kD(\omega,\omega)$,

which implies that $D(\omega,\omega)=0$ since $k\in(0,1)$ and $D(\omega,\omega)<\infty$.

For every $x\in X$, let

$\delta(D,f,x)=\sup\{D(f^i(x),f^j(x)):i,j\in N\}$.

We have the following extension of the Banach contraction principle.

Theorem 3.3

Suppose that the following conditions hold:

(i) (X,D) *is complete*;
(ii) f *is a* k*-contraction for some* $k\in(0,1)$;
(iii) *there exists* $x_0\in X$ *such that* $\delta(D,f,x_0)<\infty$.

Then $\{f_n(x_0)\}$ *converges to* $\omega\in X$, *a fixed point of* f. *Moreover, if* $\omega'\in X$ *is another fixed point of* f *such that* $D(\omega,\omega')<\infty$, *then* $\omega=\omega'$.

Proof

Let $n\in N$ $(n\geq 1)$. Since *f* is a *k*-contraction, for all $i,j\in N$, we have
$D(f^{n+i}(x_0),f^{n+j}(x_0))\leq kD(f^{n-1+i}(x_0),f^{n-1+j}(x_0))$,

which implies that
$\delta(D,f,f^n(x_0))\leq k\delta(D,f,f^{n-1}(x_0))$.

Then, for every $n\in N$, we have
$\delta(D,f,f^n(x_0))\leq k^n\delta(D,f,x_0)$.

Using the above inequality, for every $n,m\in N$, we have
$D(f^n(x_0),f^{n+m}(x_0))\leq\delta(D,f,f^n(x_0))\leq k^n\delta(D,f,x_0)$.

Since $\delta(D,f,x_0)<\infty$and $k\in(0,1)$, we obtain
$\lim_{n,m\to\infty}D(f^n(x_0),f^{n+m}(x_0))=0$,

which implies that $\{f^n(x_0)\}$ is a D-Cauchy sequence.

Since (X,D) is D-complete, there exists some $\omega\in X$ such that $\{f^n(x_0)\}$ is D-convergent to ω.
On the other hand, since *f* is a *k*-contraction, for all $n\in N$, we have
$D(f^{n+1}(x_0),f(\omega))\leq kD(f^n(x_0),\omega)$.

Letting $n\to\infty$ in the above inequality, we get
$\lim_{n\to\infty}D(f^{n+1}(x_0),f(\omega))=0$.

Then $\{f^n(x_0)\}$ is D-convergent to $f(\omega)$. By the uniqueness of the limit (see Proposition 2.4), we get $\omega=f(\omega)$, that is, ω is a fixed point of *f*.

Now, suppose that $\omega'\in X$ is a fixed point of *f* such that $D(\omega,\omega')<\infty$. Since *f* is a *k*-contraction, we have
$D(\omega,\omega')=D(f(\omega),f(\omega'))\leq kD(\omega,\omega')$,

which implies by the property (D_1) that $\omega=\omega'$.

Observe that we can replace condition (iii) in Theorem 3.3 by*(iii)′*
there exists $x_0\in X$ such that $\sup\{D(x_0,f^r(x_0)):r\in N\}<\infty$.
In fact, since *f* is a *k*-contraction, we obtain easily that
$\delta(D,f,x_0)\leq\sup\{D(x_0,f^r(x_0)):r\in N\}$.

So condition (iii)′ implies condition (iii).

The following result (see Kirk and Shahzad [6]) is an immediate consequence of Proposition 2.8and Theorem 3.3.

Corollary 3.4

Let(X,d)*be a completeb-metric space and*$f:X\to X$ *be a mapping*. *Suppose that for some*$k\in(0,1)$, *we have*
$d(f(x),f(y))\leq kd(x,y)$,for every $(x,y)\in X\times X$.

If there exists$x_0\in X$*such that*
$\sup\{d(f^i(x_0),f^j(x_0)):i,j\in N\}<\infty$,

then the sequence$\{f_n(x_0)\}$*converges to a fixed point off*. *Moreover*, *fhas one and only one fixed point*.

Note that in [1], there is a better result than this given by Corollary 3.4.

The next result is an immediate consequence of Proposition 2.10 and Theorem 3.3.

Corollary 3.5

Let (X,d) *be a complete dislocated metric space and* $f:X\to X$ *be a mapping. Suppose that for some* $k\in(0,1)$, *we have*
$d(f(x),f(y))\leq kd(x,y)$, for every $(x,y)\in X\times X$.

If there exists $x_0\in X$ *such that*
$\sup\{d(f^i(x_0),f^j(x_0)):i,j\in N\}<\infty$,

then the sequence $\{f_n(x_0)\}$ *converges to a fixed point of* f. *Moreover,* f *has one and only one fixed point.*

The following result is an immediate consequence of Proposition 2.15, and Theorem 3.3.

Corollary 3.6

Let (X_ρ,ρ) *be a complete modular space and* $f:X\to X$ *be a mapping. Suppose that for some* $k\in(0,1)$, *we have*
$\rho(f(x)-f(y))\leq k\rho(x-y)$, for every $(x,y)\in X_\rho\times X_\rho$.

Suppose also that ρ *satisfies the Fatou property. If there exists* $0\in X_\rho$ *such that*
$\sup\{\rho(f^i(x0)-f^j(x_0)):i,j\in N\}<\infty$,

then the sequence $\{f_n(x_0)\}$ ρ*-converges to some* $\omega\in X_\rho$, *a fixed point of* f. *Moreover, if* $\omega'\in X_\rho$ *is another fixed point of* f *such that* $\rho(\omega-\omega')<\infty$, *then* $\omega=\omega'$.

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