RECURRENCE RELATION FOR ACHROMATIC NUMBER OF LINE GRAPH OF GRAPH

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Abstract – In this paper, we have studied vertex coloring, chromatic and achromatic number of a graph. For certain n , upper bound of A(n) is discussed. A recurring relation is obtained for A(n), n ≥ 4.

Key Words: k-colouring, colourclass, chromatic number, achromatic number, Line graph.

1.INTRODUCTION

A k-colouring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. Often we use $S = [k] = \{1, 2, .., k\}$. The labels are colours. The vertices of one colour form a colourclass. A k-colouring is proper if adjacent vertices have different colours or labels.

A graph is k-colourable if it has a proper k-colouring. The chromatic number $\chi(G)$ is the least k, such that G is k-colourable.i.e there does not exist any proper $k - 1$ colouring.

The Achromatic number : The largest k so that there exists a complete k-colouring of V (G) is called the achromatic number $\Psi(G)$. For any k between $\chi(G)$ and $\Psi(G)$ a complete k-colouring of G exists.

Line Graph: The Line graph of G, written as L(G), is the simple graph whose vertices are the edges of G, with $ef \in E(L(G))$ when e and f have a common endpoint in G.

Let $G = K_n$, then $\Psi(G) = \Psi(K_n) = A(n)$

The achromatic number A(n) of the line graph of K n i.e. $A(n) = \Psi(L(K_n))$.

Results and Discussion:

Lemma 1. For any $t < (n - 1)/2$ $A(n) \le \max\{g(n, t + 1), h(n, t + 1)\}$

Proof. Consider any complete k- colouring of

L(K n). Assume that there is a

colourclass Γ with $s \le t$ edges of K_n in it. (i.e. 2s vertices of $L(K_n)$

Let S be the set of 2s nodes of K_n covered by the s edges in Γ.

An edge of K_n is adjacent to an edge of Γ in $L(K_n)$ has an endnode inS Since K_n is n − 1 regular, at each point of S, there are n − 1 edges of K_n incident with it. i.e.2s(n − 1) edges in K n incident on S.

Thus, there are 2s(2s − 1) posssible edges in S,but counted twice.

Therefore, total No. of edges in S is

$2s(2s - 1)]/2 = S(2s - 1)$

Thus there are s(2s − 1) edges of K_n incident with two points of S.

Thus No. of edges of K_n not in Γ but incident with a point of S is $2s(n - 1) - s(2s - 1) - s$

$= 2sn - 2s - 2s^2 + s - s$

$= 2s(n - s - 1)$

$= g(n, s)$

Now, Γ is a colourclass and k-colouring is a proper colouring.

Therefore, Γ must be adjacent to atleast one edge of every other colourclass.

Thus, there are atleast k −1 edges starting from S but having other endpoints outside Γ and S There are such g(n, s) edges.

$k - 1 \le g(n, s)$

$k \le g(n, s) + 1$

But, $s \le t$ and $t < (n - 1)/2$ and g(n, s) is increasing for $t < (n - 1)/2$

$g(n, s) + 1 \le g(n, t) + 1$

$k \le g(n, t) + 1$

Hence, $A(n) \le \max\{g(n, t) + 1, h(n, t) + 1\}$ as if $g(n, t) + 1$ is maximum $A(n) = k \le g(n, t) + 1$. if $g(n, t) + 1$ is not maximum then $g(n, t) + 1 \le h(n, t) + 1$

But $k = A(n) \le g(n, t) + 1 \le h(n, t) + 1$

Thus, $A(n) \le \max\{g(n, t) + 1, h(n, t) + 1\}$

Now if no colourclass contains No. of edges ≤ t i.e. every colourclass contains atleast t + 1 edges.

But there are in all n(n − 1)/2 edges,

We can have almost $[(n[n-1]/2)/(t+1)]$ colourclasses. i.e. $[n(n-1)]/(2(t+1) = h(n, t + 1)$ Thus, $k \le h(n, t + 1)$

Thus by similar arguements,

$A(n) = k \le \max\{g(n, t) + 1, h(n, t) + 1\}$

Notations:
Let [x] denote the greatest integer in x.
$\beta_t(n) = \max\{g(n, t) + 1, [h(n, t + 1)]\}$
$B(n) = \min\{\beta_t(n) | 0 < t < (n-1)/2$
Lemma can be formulated as $A(n) \le B(n)$

Lemma 2. suppose $t \ge 2$, $4t^2 - t \le n \le 4t^2 + 3t - 1$ then $B(n) = g(n, t) + 1$, If $4t^2 + 3t \le n \le 4(t + 1)^2 - t - 2$ then $B(n) = [h(n, t + 1)]$.
Proof. We need to compare g and h. Note that g is an integer.

So, $g(n, t) + 1 \le [h(n, t)]$ iff $g(n, t) + 1 \le h(n, t)$

i.e. iff $0 \le h(n, t) - g(n, t) - 1$

Let $p(n, t) = h(n, t) - g(n, t) - 1$

Thus $g(n, t) + 1 \le [h(n, t)]$ iff $p(n, t) \ge 0$ **.......(1)**

$p(n, t) = h(n, t) - g(n, t) - 1$

$=[n(n - 1)]/2t - 2t(n - t - 1) - 1$

$=[n(n - 1) - 4t^2(n - t - 1) - 2t]/2t$
Now $2tp(n, t) = n^2 - n - 4t^2 n + 4t^3 + 4t^2 - 2t$

$= n^2 - (4t^2 + 1)n + 4t^3 + 4t^2 - 2t$
Now consider $q(n, t) = h(n, t + 1) - g(n, t) - 1$

Thus, $g(n, t) + 1 \le [h(n, t + 1)]$
iff $q(n, t) \ge 0$ **.........(2)**

$q(n, t) = h(n, t + 1) - g(n, t) - 1$

$=[n(n - 1)]/[2(t + 1)] - 2t(n - t - 1) - 1$

Now
$2(t + 1)q(n, t) = n(n - 1) - 4(t^2 + t)(n - t - 1) - 2(t + 1)$

$= n^2 - n4nt^2 + 4t^3 + 4t^2 - 4nt + 4t^2 + 4t - 2(t + 1)$

$= n^2 - (4t^2 + 4t + 1)n + 4t(t + 1)^2 - 2(t + 1)$

Now consider,

$p(4t^2 - t - 1, t) = -3t^2 + t + 2 < 0$ if $t > 0$

$p(4t^2 - t, t) = t^2 - t \ge 0$ if $t > 0$

$q(4t^2 - t, t) = - - 3t^2 - 3t < 0$ if $t > 0$

$q(4t^2 + 3t, t) = t^2 - t - 2 \ge 0$ if $t \ge 2$

Let us denote this set of statement as $*\Lambda$ Now, let $t \ge 2$, differentiating $q(x, t)$ with

respect to x

$D_x q(x, t) = 2x - (4t^2 + 4t + 1)$

Consider $4t^2 - t \le x \le 4t^2 + 3t - 1$

For $x \ge 4t^2 - t$

$D_x q(x, t) = 2(4t^2 - t) - (4t^2 + 4t + 1)$

$= 8t^2 - 2t - 4t^2 - 4t - 1$

$= 4t^2 - 6t - 1$

≥ 0

$(t \ge 2)$

Thus, for $x \ge 4t^2 - t$ we get $D_x q(x, t) > 0$

Therefore, $q(x, t)$ is increasing function. By Λ, $q(4t^2 + 3t - 1) < 0$

Hence $q(n, t) < 0$

$\forall n$ such that $4t^2 - t \le n \le 4t^2 + 3t - 1$

By (2) ,
for such n, $h(n, t + 1) < g(n, t) + 1$ **..........(3)**

So, $\beta_t(n) = g(n, t) + 1$ **..............(4)**

by definition of $\beta_t(n)$

Now,if $t < u < (n-1)/2$,then $g(n, u) + 1 \ge g(n, t) + 1$ (since g is increasing.)

$> h(n, t + 1)$ **by 3**

$\ge h(n, u + 1)$(as g is increasing f or $y < (x - 1)/2$ and h is decreasing).
Thus,
$\beta_u(n) \ge \beta_t(n)$ **...............(B)**

Now, consider $s < t$, differentiating $p(x, t)$ with respect to x from (1)

$D_x p(x, t) = 2x - (4t^2 + 1)$

For $x \geq 4t^2 - t$

$D_x p(x, t) = 2(4t^2 - t) - 4t^2 - 1$

$= 8t^2 - 2t - 4t^2 - 1$
$= 4t^2 - 2t - 1$

> 0 as $t \geq 2$
Hence, p(x, t) is increasing.

By Λ, $P(4t^2 - t, t) \geq 0$

So, $P(n, t) \geq 0$

$\forall n \geq 4t^2 - t$

Thus **by (1)** for such n,

$[h(n, t)] \geq g(n, t) + 1$ **..............(5)**

(as $t \geq 2$)2.

Hence, for s < t, we get,

$\beta_s(n) \geq [h(n, s + 1)] \geq [$
$h(n, t)$
$]$ (as h is increasing)

$\geq g(n, t) + 1$ **by(5)**

$= \beta_t(n)$

Therefore, if s < t then $\beta_s(n) \geq \beta_t(n)$
..............(A)
Thus, $B(n) = \beta_t(n) = g(n, t) +$
1 as desired.

By (A) and (B) we get

$B(n) = \min\{\beta_t(m)\}$

$= \beta_t(n)$

$= g(n, t) + 1$

Now,l
et us consider $4t^2 + 3t \leq n \leq 4(t + 1)^2 - t - 2$

We know that, forx $\geq 4t^2 - t$, q(x, t) is increasing.

By Λq$(4t^2 + 3t, t) > 0$

It implies $q(n, t) > 0$,

for above choice of n

So,**by (2)**

$[h(n, t + 1)] \geq g(n, t) + 1$ **..........(6)**
for $n \geq 4t^2 + 3t$

Hence,$\beta^t(n) = [h(n, t + 1)]$ **.........(7)**
for $n \geq 4t^2 + 3t$

Now if s < t

$\beta_s(n) \geq [h(n, s + 1)] \geq [h(n, t + 1)] = \beta_t(n)$

For s < t, $\beta_s(n) \leq \beta_t(n)$ **...........(C)**

Consider $t + 1 \leq u <$
$(n-1)*2$

$p(x, t + 1) = x^2 - (4(t + 1)^2 + 1)x + 4(t + 1)^3 + 4(t + 1)^2 - 2(t + 1)$
Therefore,

$D_x p(x, t + 1) = 2x - 4(t + 1)^2 - 1$

$D_n p(n, t + 1) = 2n - 4(t + 1)^2 - 1$

If $n \geq 4t^2 + 3t$

$D_n p(n, t + 1) = 2(4t^2 + 3t) - 4(t + 1)^2 - 1$

$= 8t^2 + 6t - 4t^2 - 8t - 5$

$= 4t^2 - 2t - 5$

≥ 0
$\forall t \geq 2$

Thus, $D_n p(n, t + 1)$ is positive for $n \geq 4t^2 + 3t$

Hence p is increasing for $n \geq 4t^2 + 3t$.

$p(4(t + 1)^2 - (t + 1) - 1, t + 1) < 0$

$p(n, t + 1) < 0$

byΛ

$\forall n \in (4t^2 + 3t, 4(t + 1)^2 - t - 2)$

Hence for n in this range, $[h(n, t + 1)] \geq g(n, t + 1) + 1$
...........(8)

Thus,

$\beta t(n) =$
$[h(n, t + 1)$
$] \leq g(n, t + 1) + 1$

$\leq g(n, u) + 1$

by(7) and by(8)

(since g is increasing.)

$\leq \beta_u (n)$

Thus,

$\beta_t(n) \leq \beta_u (n)$

for $t + 1 < u <$
$(n - 1$
$)/2$
........(D)

Hence by **(C)and(D)**, $\beta_t (n) = \min\{\beta_t (n)\}$

Therefore, $B(n) = \beta_t (n) = [h(n, t + 1)$
$]$ **by(7)**

Thus proved.

Theorem. $A(n + 2) \geq A(n) + 2$
if $n > 4$.

Proof. Consider an optimal colouring of K_n i.e. with $A(n)$ colours. We select a maximal collection of Γ disjoint edges of different colours i.e. no colour is repeated in Γ i.e. there can be maximum one edge of each colour. But in order to collect only disjoint edges, there may not be any edge of a colour. But a vertex at which such an edge is incident is present in Γ with some edge of different colour.
Thus Γ meets every colourclass, Γ is a matching and can have maximum n /2 edges since there are n vertices and edeges are disjoint. Let st be an edge of Γ .

There are n − 2 more edges incidents at t other than st, and these are all of distinct colours being a proper colouring.

Let tu be the edge whose colour does not occur in Γ .
We collect a maximal set Δ of disjoint edges coloured with colours not used in Γ . tu
is one of edge in Δ.Now, we prove that subgraph G generated by disjoint Γ ∪ Δ is
bipartite.

Let there be an odd cycle in G = Γ ∪ Δ, say $c = e_1 \; e_2 \; e_3 \ldots e_{2n} \, e_{2n+1}$
Let $e_1 \in \Delta$, w.l.o.g. then since Δ is disjoint edge's collection e_2 . which is having one
vertex common with e_1 can not be in Δ.

Therefore $e_2 \in \Gamma$. By similar argument $e_3 \in \Delta$

and so on . . .
Thus, $e_{2n} \in \Gamma$

Therefore, $e_{2n+1} \in \Delta$ but e_{2n+1} and $e_1 \in \Delta$ and have a vertex in common.

A contradiction to the construction of Δ.

Hence,there is no odd cycle in G, giving G is bipartite.

So,there exist a proper 2 colouring of vertices of G. (i.e. a vertex colouring)
Let vertices in one partition of G be coloured by black and vertices in another partition

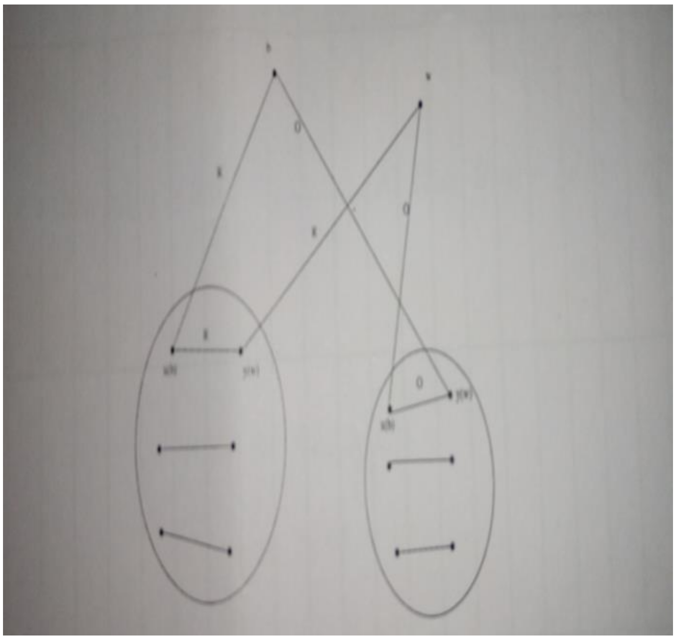
of G be coloured by white.

Now, we add two nodes b and w in K_n. Let xy be an edge of G such that x is black

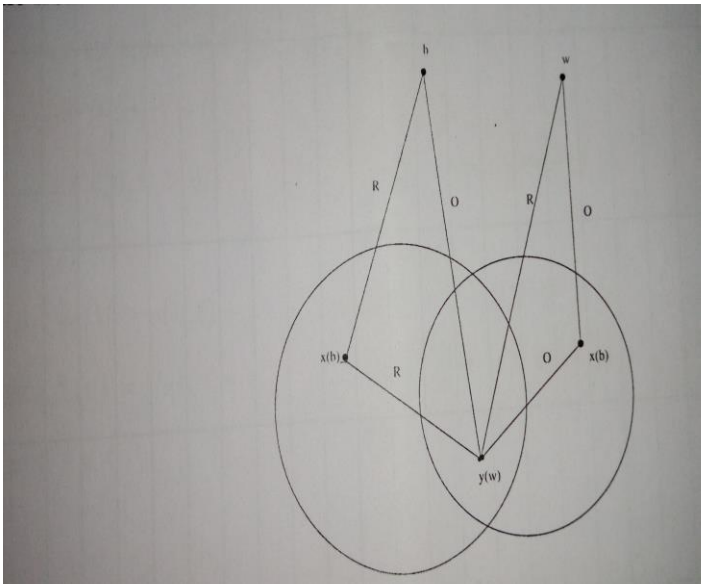
and y is white. xy either is in Γ or in Δ.

If xy ∈ Γ , we colour edges bx and wy both with the same colour as that of xy. If

xy ∈ Δ,we colour edges wx and by both with the same colour as that of xy



Now consider any vertex of G. Since there are at most two edges, one from and one from Γ at each node of G this is consistent colouring. See below.



Look at y. There is one orange colour edge yb and one red color edge yw. (This has become possible to get two different colour edges at a common vertex of Γ and is Δ due to bipartition) The edges through y, orange and red inside Γ and Δ are going to be of some different colours later on. More over, since all the colours in Γ ∪ Δ are different, no colour appears twice at either b or w.
Now, we erase the old colours on the edges of Γ and make Γ a new colourclass using colour other than used A(n) colours. Note here that for any edge xy ∈ Γ .Also, bx

and wy are of same old colour. x and y still belong in the support of old coloursclass.
Infact b and w also are in the support of old colourclass in this new colouring.
Now, erase the old colours on the edges of Δ and make Δ (Λ) = Δ ∪ {bw} a new colourclass i.e. other than used Δ(n) + 1 colours. Similar arguments hold true even for each each xy ∈ Δ Thus, this new colouring with A(n) + 2 colours is a proper colouring. Since the supports of old colours are either left the same or enlarged by b and w, it follows that any two old colourclasses still meet. Also, Γ meets every old colour. Therfore, (A(n) + 1)st colour meets every colourclass. Δ(Λ)meets every old colour not appering on an edge in Γ . bw is coloured in (A(n) + 2)th colour. Hence (A(n) + 2)th meets every old colour,as bw meets all colours and (A(n) + 1)st (A(n) + 2)th colours meet on the special edges st and tu. Thus colouring is complete.
But this is a colouring of G, a subgraph of K_{n+2} , Thus, If Ψ(G) ≥ A(n) + 2 then Ψ(L(K n+2)) = A(n + 2) ≥ A(n) + 2
Therefore, A(n + 2) ≥ A(n) + 2.

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