Application of Homotopy Analysis Method for Solving various types of Problems of Partial Differential Equations

V.P.Gohil[1], Dr. G. A. Ranabhatt[2]

[1,2]*Assistant Professor, Department of Mathematics, Government Engineering College, Bhavnagar, India*

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Abstract - *In this paper, various types of linear, non-linear, homogeneous, non homogeneous problems of partial differential equations discussed. Also shown that homotopy analysis method applied successfully for solving non homogeneous and non linear equations*

Key Words: **homotopy analysis method, partial differential equation, linear, homogeneous, linear, non linear, homogeneous, non homogeneous**

1.INTRODUCTION

In recent years, this method (HAM) has been successfully employed to solve many types of non linear, homogeneous or non homogeneous, equations and systems of equations as well as problems in science and engineering . Very recently, Ahmad Bataineh et al.([2]) presented two modi_cations of HAM to solve linear and non linear ODEs. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called h -curve, it is easy to determine the valid regions of h to gain a convergent series solution. Thus, through HAM, explicit analytic solutions of non linear problems are possible.
Systems of partial differential equations (PDEs) arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of the chemical reaction-diffusion model. Very recently, Batiha et al. [2] improved Wazwazs [9] results on the application of the variational iteration method (VIM) to solve some linear and non linear systems of PDEs. In [8], Saha Ray implemented the modified Adomian decomposition method (ADM) for solving the coupled sine-Gordon equation.

2. HOMOTOPY ANALYSIS METHOD

We consider the following differential equations,

$N_i[S_i(x,t)] = 0, i = 1,2,\dots,n$

Where N_i are nonlinear operators that the represents the whole equations, x and t are independent variables and $S_i(x,t)$ are unknown functions respectively.

By means of generalizing the traditional homotopy method, Liao constructed the so-called zero-order deformation equations

$$(1-q)L[\emptyset_i(x,t,;q) - S_{i,0}(x,t)] = qh_iN_i[\emptyset_i(x,t,;q)] \tag{1}$$

Where $q \in [0,1]$ is an embedding operators, h_i are nonzero auxiliary functions, L is an auxiliary linear operator, $S_{i,0}(x,t)$ are initial guesses of $S_i(x,t)$ and $\emptyset_i(x,t,;q)$ are unknown functions.

It is important to note that, one has great freedom to choose auxiliary objects such as h_i and L in HAM.

When $q = 0$ *and* $q = 1$ we get by (1),

$\emptyset_i(x,t,;0) = S_{i,0}(x,t)$ *and* $\emptyset_i(x,t,;1) = S_i(x,t)$

Thus q increase from 0 to 1, the solutions $\emptyset_i(x,t,;q)$ varies from initial guesses $S_{i,0}(x,t)$ to $S_{i,0}(x,t)$.

Expanding $\emptyset_i(x,t,;q)$ in Taylor series with respect to ,

$$\emptyset_i(x,t,;q) = S_{i,0}(x,t) + \sum_{m=1}^{\infty} S_{i,m}(x,t).q^m \tag{2}$$

Where

$$S_{i,m}(x,t) = \left[\frac{1}{m!}.\frac{\partial^m \emptyset_i(x,t,;q)}{\partial q^m}\right]_{q=0}, \tag{3}$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h_i and auxiliary functions are properly chosen than the series equtions (2) converges at $= 1$.

$$\emptyset_i(x,t,;1) = S_{i,0}(x,t) + \sum_{m=1}^{\infty} S_{i,m}(x,t) \tag{4}$$

This must be one of solutions of the original nonlinear equations.

According to (3), the governing equations can be deduced from the zero-order deformation equations (1).

Define the vectors

$$\overrightarrow{S_{i,n}} = \{S_{i,0}(x,t), S_{i,1}(x,t), S_{i,2}(x,t), \ldots, S_{i,n}(x,t)\}$$

Differentiating (1) m times with respect to the embedding parameter q and the setting $q = 0$ and finally dividing them by $m!$.

We have the so-called m^{th} order deformation equations

$$L[S_{i,m}(x,t) - \chi_m S_{i,m-1}(x,t)] = h_i R_{i,m}(\overrightarrow{S_{i,m-1}}) \quad (5)$$

(5)

Where

$$R_{i,m}(S_{i,m-1}) = \left[\frac{1}{(m-1)!} \cdot \frac{\partial^{m-1} N_i[\emptyset_i(x,t;q)]}{\partial q^{m-1}}\right]_{q=0} \quad (6)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

2. HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION

Consider homogeneous linear differential equation

$$u_t + u_x + u = 0 \quad (7)$$

Subject to the initial condition

$$u(x,0) = e^x \quad (8)$$

To solve this system (2.57) to (2.58) by HAM, first we choose initial approximation

$$u_0(x,t) = e^x$$

And the linear operator

$$L(\phi_1(x,\ t;q)) = \frac{\partial \phi_1(x,\ t;q)}{\partial x}$$

With the property $L(C) = 0$ where C is integral constant.

We define system of non-linear operator as

$$N(\phi_1(x,\ t;q)) = \frac{\partial^2 \phi_1(x,\ t;q)}{\partial x^2} + \frac{\partial \phi_1(x,\ t;q)}{\partial x} + \phi_1(x,\ t;q) \quad (9)$$

Using the above definition, we construct the zeroth-order deformation equations

$$(1-q)[\phi_1(x,\ t;q) - S_1(x,\ t)] = qhN(\phi_1(x,\ t;q)) \quad (10)$$

Obviously, when $q = 0$ and $q = 1$ we get

$$\phi_1(x,\ t;q) = S_{1,0}(x,\ t) = u_0(x,\ t) \text{ and}$$

$$\phi_1(x,\ t;1) = u(x,\ t) \quad (11)$$

As q increase 0 to 1, ϕ_1 varies from $u_0(x,\ t)$ to $u(x,\ t)$ Expanding $\phi_1(x,\ t;q)$ in Taylor series with respect to q,

$$\phi_1(x,\ t;q) = S_{1,0}(x,\ t) + \sum_{m=1}^{\infty} S_{1,m}(x,\ t) \cdot q^m \quad (22)$$

Where

$$S_{1,m}(x,\ t) = \left[\frac{1}{m!} \frac{\partial^m \phi_1(x,\ t;q)}{\partial q^m}\right]_{q=0} \quad (23)$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h and auxiliary functions are properly chosen than the series equation (2.63) converges at $q = 1$.

$$\phi_1(x,\ t;1) = S_{1,0}(x,\ t) + \sum_{m=1}^{\infty} S_{1,m}(x,\ t)$$

i.e. $u(x,\ t) = S_{1,0}(x,\ t) + \sum_{m=1}^{\infty} S_{1,m}(x,\ t)$

This must be one of solutions of the original non linear equations as proved by Liao Define the vectors

$$\overrightarrow{S_{1,n}} = (S_{1,0}(x,\ t),\ S_{1,1}(x,\ t),\ S_{1,2}(x,\ t),\ \ldots\ldots S_{1,n}(x,\ t)) \quad (24)$$

We have the so-called m^{th} order deformation equations

$$L[S_{1,m}(x,\ t) - \chi_{1,m} S_{1,m}(x,\ t)] = hR_{1,m}(\overrightarrow{S_{1,m-1}}) \quad (25)$$

Where

$$R_{1,m}(\overrightarrow{S_{1,m-1}}) = \left[\frac{1}{(m-1)!} \frac{\partial^{m-1} \phi_1(x,\ t;q)}{\partial q^{m-1}}\right]_{q=0} \quad (26)$$

i.e. $R_{1,m}\overrightarrow{S_{1,m-1}} = (S_{1,m-1})_t + S_{1,m-1}(S_{1,m-1})_x + S_{1,m-1}$ (27)

$$S_{1,m}(x,\ t)=\chi_m S_{1,m-1}(x,\ t)+h\int_0^t R_{1,m}\left(\overrightarrow{S_{1,m-1}}\right)\ dt+c \tag{28}$$

Now we will calculate

$$S_{1,1}(x,\ t)=\chi_m S_{1,0}(x,\ t)+h\int_0^t R_{1,1}\left(\overrightarrow{S_{1,0}}\right)\ dt \tag{29}$$

Where

$R_{1,1}\left(\overrightarrow{S_{1,0}}\right)=2e^x$

So

$S_{1,1}(x,\ t)=2hte^x$

Now The N^{th} order approximation can be expressed by

$$S(x,\ t)=S_{1,0}(x,\ t)+\sum_{m=1}^{N-1}S_{1,m}(x,\ t) \tag{30}$$

As $N\to\infty$ we get $S(x,\ t)\to u(x,\ t)$ with some appropriate assumption of h

3. NON HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION

Consider non homogeneous linear differential equation

$$u_t+u_x+u-1=0 \tag{31}$$

Subject to the initial condition

$$u(x,0)=e^x \tag{32}$$

To solve this system (31) to (32) by HAM, first we choose initial approximation

$u_0(x,t)=e^x$

And the linear operator

$$L(\phi_1(x,\ t\ ;q))=\frac{\partial\phi_1(x,\ t\ ;q)}{\partial x}$$

With the property $L(C)=0$ where C is integral constant.

We define system of non-linear operator as

$$N(\phi_1(x,\ t\ ;q))=\frac{\partial^2\phi_1(x,\ t\ ;q)}{\partial x^2}+\frac{\partial\phi_1(x,\ t\ ;q)}{\partial x}+\phi_1(x,\ t\ ;q)-1 \tag{33}$$

Using the above definition, we construct the zeroth-order deformation equations

$$(1-q)\left[\phi_1(x,\ t\ ;q)-S_1(x,\ t)\right]=qhN(\phi_1(x,\ t\ ;q)) \tag{34}$$

Obviously, when $q=0$ and $q=1$ we get

$\phi_1(x,\ t\ ;q)=S_{1,0}(x,\ t)=u_0(x,\ t)$ and

$$\phi_1(x,\ t\ ;1)=u(x,\ t) \tag{35}$$

As q increase 0 to 1, ϕ_1 varies from $u_0(x,\ t)$ to $u(x,\ t)$

Expanding $\phi_1(x,\ t\ ;q)$ in Taylor series with respect to q ,

$$\phi_1(x,\ t\ ;q)=S_{1,0}(x,\ t)+\sum_{m=1}^{\infty}S_{1,m}(x,\ t)\cdot q^m \tag{36}$$

Where

$$S_{1,m}(x,\ t)=\left[\frac{1}{m!}\frac{\partial^m\phi_1(x,\ t\ ;q)}{\partial q^m}\right]_{q=0} \tag{37}$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h and auxiliary functions are properly chosen than the series equation (2.76) converges at $q=1$.

$$\phi_1(x,\ t\ ;1)=S_{1,0}(x,\ t)+\sum_{m=1}^{\infty}S_{1,m}(x,\ t)$$

i.e. $u(x,\ t)=S_{1,0}(x,\ t)+\sum_{m=1}^{\infty}S_{1,m}(x,\ t)$

This must be one of solutions of the original non linear equations as proved by Liao Define the vectors

$$\overrightarrow{S_{1,n}}=\left(S_{1,0}(x,\ t),\ S_{1,1}(x,\ t),\ S_{1,2}(x,\ t),\ .....S_{1,n}(x,\ t)\right) \tag{38}$$

We have the so-called m^{th} order deformation equations

$$L\left[S_{1,m}(x,\ t)-\chi_m S_{1,m}(x,\ t)\right]=hR_{1,m}\left(\overrightarrow{S_{1,m-1}}\right) \tag{39}$$

Where

$$R_{1,m}\left(\overrightarrow{S_{1,m-1}}\right)=\left[\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi_1(x,\ t\ ;q)}{\partial q^{m-1}}\right]_{q=0} \tag{40}$$

i.e. $$R_{1,m}\overrightarrow{S_{1,m-1}}=\left(S_{1,m-1}\right)_t+S_{1,m-1}\left(S_{1,m-1}\right)_x+S_{1,m-1} \tag{41}$$

$$S_{1,m}(x,\ t)=\chi_m S_{1,m-1}(x,\ t)+h\int_0^t R_{1,m}\left(\overrightarrow{S_{1,m-1}}\right)\ dt+c \tag{42}$$

Now we will calculate

$$S_{1,1}(x,\ t)=\chi_m S_{1,0}(x,\ t)+h\int_0^t R_{1,1}\left(\overrightarrow{S_{1,0}}\right)\ dt \tag{43}$$

Where

$$R_{1,1}\left(\overrightarrow{S_{1,0}}\right)=2e^{x}-1$$

So

$$S_{1,1}(x,\ t)=h\left[2te^{x}-t\right]$$

Now the N^{th} order approximation can be expressed by

$$S(x,\ t)=S_{1,0}(x,\ t)+\sum_{m=1}^{N-1}S_{1,m}(x,\ t) \tag{44}$$

As $N\to\infty$ we get $S(x,\ t)\to u(x,\ t)$ with some appropriate assumption of h

4. NON HOMOGENEOUS NON LINEAR PARTIAL DIFFERENTIAL EQUATION

Consider non homogeneous non linear differential equation

$$u_t+u\cdot u_x+u-1=0 \tag{45}$$

Subject to the initial condition

$$u(x,0)=e^{x} \tag{46}$$

To solve this system (45) to (46) by HAM, first we choose initial approximation

$$u_0(x,t)=e^{x}$$

And the linear operator

$$L(\phi_1(x,\ t\ ;q))=\frac{\partial\phi_1(x,\ t\ ;q)}{\partial x}$$

With the property $L(C)=0$ where C is integral constant.

We define system of non-linear operator as

$$N(\phi_1(x,\ t\ ;q))=\frac{\partial^2\phi_1(x,\ t\ ;q)}{\partial x^2}+\phi_1(x,\ t\ ;q)\frac{\partial\phi_1(x,\ t\ ;q)}{\partial x}+\phi_1(x,\ t\ ;q)-1 \tag{47}$$

Using the above definition, we construct the zeroth-order deformation equations

$$(1-q)\left[\phi_1(x,\ t\ ;q)-S_1(x,\ t)\right]=qhN(\phi_1(x,\ t\ ;q)) \tag{48}$$

Obviously, when $q=0$ and $q=1$ we get

$$\phi_1(x,\ t\ ;q)=S_{1,0}(x,\ t)=u_0(x,\ t) \quad \text{and}$$

$$\phi_1(x,\ t\ ;1)=u(x,\ t) \tag{49}$$

As q increase 0 to 1, ϕ_1 varies from $u_0(x,\ t)$ to $u(x,\ t)$

Expanding $\phi_1(x,\ t\ ;q)$ in Taylor series with respect to q,

$$\phi_1(x,\ t\ ;q)=S_{1,0}(x,\ t)+\sum_{m=1}^{\infty}S_{1,m}(x,\ t)\cdot q^{m} \tag{50}$$

Where

$$S_{1,m}(x,\ t)=\left[\frac{1}{m!}\frac{\partial^m\phi_1(x,\ t\ ;q)}{\partial q^m}\right]_{q=0} \tag{51}$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h and auxiliary functions are properly chosen than the series equation (2.90) converges at $q=1$.

$$\phi_1(x,\ t\ ;1)=S_{1,0}(x,\ t)+\sum_{m=1}^{\infty}S_{1,m}(x,\ t)$$

i.e. $u(x,\ t)=S_{1,0}(x,\ t)+\sum_{m=1}^{\infty}S_{1,m}(x,\ t)$

This must be one of solutions of the original non linear equations as proved by Liao Define the vectors

$$\overrightarrow{S_{1,n}}=\left(S_{1,0}(x,\ t),\ S_{1,1}(x,\ t),\ S_{1,2}(x,\ t),\ .....S_{1,n}(x,\ t)\right) \tag{52}$$

We have the so-called m^{th} order deformation equations

$$L\left[S_{1,m}(x,\ t)-\chi_m S_{1,m}(x,\ t)\right]=hR_{1,m}\left(\overrightarrow{S_{1,m-1}}\right) \tag{53}$$

Where

$$R_{1,m}\left(\overrightarrow{S_{1,m-1}}\right)=\left[\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi_1(x,\ t\ ;q)}{\partial q^{m-1}}\right]_{q=0} \tag{54}$$

i.e. $R_{1,m}\overrightarrow{S_{1,m-1}}=\left(S_{1,m-1}\right)_t+S_{1,m-1}\left(S_{1,m-1}\right)_x+S_{1,m-1}-1$ (55)

$$S_{1,m}(x,\ t)=\chi_m S_{1,m-1}(x,\ t)+h\int_0^t R_{1,m}\left(\overrightarrow{S_{1,m-1}}\right)\ dt+c \tag{56}$$

Now we will calculate

$$S_{1,1}(x,\ t)=\chi_m S_{1,0}(x,\ t)+h\int_0^t R_{1,1}\left(\overrightarrow{S_{1,0}}\right)\ dt \tag{57}$$

Where

$$R_{1,1}\left(\overrightarrow{S_{1,0}}\right)=e^{2x}+e^{x}-1$$

So

$$S_{1,1}(x,\ t)=ht\left(e^{2x}+e^{x}-1\right)$$

Now the N^{th} order approximation can be expressed by

$$S(x,\ t)=S_{1,0}(x,\ t)+\sum_{m=1}^{N-1}S_{1,m}(x,\ t) \tag{58}$$

As $N\to\infty$ we get $S(x,\ t)\to u(x,\ t)$ with some appropriate assumption of h

5. HOMOGENEOUS NON LINEAR PARTIAL DIFFERENTIAL EQUATION

Consider homogeneous non linear differential equation

$$u_t + u \cdot u_x + u = 0 \tag{60}$$

Subject to the initial condition

$$u(x,0) = e^x \tag{60}$$

To solve this system (59) to (60) by HAM, first we choose initial approximation

$$u_0(x,t) = e^x$$

And the linear operator

$$L\left(\phi_1(x,\ t;q)\right) = \frac{\partial \phi_1(x,\ t;q)}{\partial x}$$

With the property $L(C) = 0$ where C is integral constant.

We define system of non-linear operator as

$$N\left(\phi_1(x,\ t;q)\right) = \frac{\partial^2 \phi_1(x,\ t;q)}{\partial x^2} + \phi_1(x,\ t;q)\frac{\partial \phi_1(x,\ t;q)}{\partial x} + \phi_1(x,\ t;q)$$

(2. 101)

Using the above definition, we construct the zeroth-order deformation equations

$$(1-q)\left[\phi_1(x,\ t;q) - S_1(x,\ t)\right] = qhN\left(\phi_1(x,\ t;q)\right) \tag{61}$$

Obviously, when $q=0$ and $q=1$ we get

$$\phi_1(x,\ t;q) = S_{1,0}(x,\ t) = u_0(x,\ t) \quad \text{and}$$

$$\phi_1(x,\ t;1) = u(x,\ t) \tag{62}$$

As q increase 0 to 1, ϕ_1 varies from $u_0(x,\ t)$ to $u(x,\ t)$

Expanding $\phi_1(x,\ t;q)$ in Taylor series with respect to q ,

$$\phi_1(x,\ t;q) = S_{1,0}(x,\ t) + \sum_{m=1}^{\infty} S_{1,m}(x,\ t)\cdot q^m \tag{63}$$

Where

$$S_{1,m}(x,\ t) = \left[\frac{1}{m!}\frac{\partial^m \phi_1(x,\ t;q)}{\partial q^m}\right]_{q=0} \tag{64}$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h and auxiliary functions are properly chosen than the series equation (2.90) converges at $q=1$.

$$\phi_1(x,\ t;1) = S_{1,0}(x,\ t) + \sum_{m=1}^{\infty} S_{1,m}(x,\ t)$$

i.e. $u(x,\ t) = S_{1,0}(x,\ t) + \sum_{m=1}^{\infty} S_{1,m}(x,\ t)$

This must be one of solutions of the original non linear equations as proved by Liao Define the vectors

$$\overrightarrow{S_{1,n}} = \left(S_{1,0}(x,\ t),\ S_{1,1}(x,\ t),\ S_{1,2}(x,\ t),\ \ldots.. S_{1,n}(x,\ t)\right) \tag{65}$$

We have the so-called m^{th} order deformation equations

$$L\left[S_{1,m}(x,\ t) - \chi_m S_{1,m}(x,\ t)\right] = hR_{1,m}\left(\overrightarrow{S_{1,m-1}}\right) \tag{66}$$

Where

$$R_{1,m}\left(\overrightarrow{S_{1,m-1}}\right) = \left[\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi_1(x,\ t;q)}{\partial q^{m-1}}\right]_{q=0} \tag{67}$$

i.e. $R_{1,m}\overrightarrow{S_{1,m-1}} = \left(S_{1,m-1}\right)_t + S_{1,m-1}\left(S_{1,m-1}\right)_x + S_{1,m-1} - 1$ (68)

$$S_{1,m}(x,\ t) = \chi_m S_{1,m-1}(x,\ t) + h\int_0^t R_{1,m}\left(\overrightarrow{S_{1,m-1}}\right)\ dt + c \tag{69}$$

Now we will calculate

$$S_{1,1}(x,\ t) = \chi_m S_{1,0}(x,\ t) + h\int_0^t R_{1,1}\left(\overrightarrow{S_{1,0}}\right)\ dt \tag{70}$$

Where

$$R_{1,1}\left(\overrightarrow{S_{1,0}}\right) = e^{2x} + e^x$$

So

$$S_{1,1}(x,\ t) = ht\left(e^{2x} + e^x\right)$$

Now the N^{th} order approximation can be expressed by

$$S(x,\ t) = S_{1,0}(x,\ t) + \sum_{m=1}^{N-1} S_{1,m}(x,\ t) \tag{71}$$

As $N \to \infty$ we get $S(x,\ t) \to u(x,\ t)$ with some appropriate assumption of h

3. CONCLUSIONS

Various types of homogeneous, non homogeneous, linear, non linear partial differential equations can be solved easily by using homotopy analysis method.

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