INTEGRAL SOLUTIONS OF THE TERNARY CUBIC EQUATION

$$3(x^2+y^2)-4xy+2(x+y+1)=972z^3$$

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Abstract: *The non-homogeneous cubic equation with three unknowns represented by the Diophantine equation* $3(x^2+y^2)-4xy+2(x+y+1)=972\,z^3$ *is analyzed for its patterns of non – zero integral solutions. A few interesting properties among the solutions and some special polygonal numbers are presented.*

Keywords: Cubic Equation with Three Unknowns, Integral solutions

Introduction:

Mathematics is the language of patterns and relationships and is used to describe anything that can be quantified. Diophantine equations has been matter of interest to various mathematicians. The problem of finding all integer solutions of a diophantine equation with three or more variables and equations of degree at least three, in general presents a good deal of difficulties. In [1-3], theory of numbers were discussed. In [4,5], a special Pythagorean triangle problem have been discussed for its integral solutions. In [6-10], higher order equations are considered for integral solutions.

In this communication, the non-homogeneous cubic equation with three unknowns represented by the equation $3(x^2+y^2)-4xy+2(x+y+1)=972\,z^3$ is considered and in particular a few interesting relations among the solutions are presented.

Notations:

Obl_n = Oblong number of rank 'n'.

$T_{m,n}$ = Polygonal number of rank 'n' with sides 'm'.

CS_n = Centered Square number of rank 'n'.

SO_n = Stella octangula number of rank 'n'.

O_n = Octahedral number of rank 'n'.

Gno_n = Gnomonic number of rank 'n'.

$Star_n$ = Star number of rank 'n'.

TO_n = Truncated octahedral number of rank 'n'.

P_n^m = Pyramidal number of rank 'n' with sides 'm'.

Method of Analysis:

The Cubic equation to be solved for its non-zero integral solution is

$$3(x^2+y^2)-4xy+2(x+y+1)=972z^3 \tag{1}$$

On substitution of the transformations,

$$x=u+v\ ,\ y=u-v \tag{2}$$

in (1) leads to,

$$(u+1)^2+5v^2=486z^3 \tag{3}$$

We illustrate below four different patterns of non-zero distinct integer solutions to (1).

Pattern: 1

Assume $z=z(a,b)=a^2+5b^2$ (4)

where a and b are non-zero integers.

and write $486=(19+5i\sqrt{5})(19-5i\sqrt{5})$ (5)

Substituting (4) & (5) in (3), and using factorization method,

$$\left((u+1)+i\sqrt{5}v\right)\left((u+1)-i\sqrt{5}v\right)=(19+5i\sqrt{5})(19-5i\sqrt{5})\left((a+i\sqrt{5}b)^3\,(a-i\sqrt{5}b)^3\right) \tag{6}$$

Equating the like terms and comparing real and imaginary parts, we get

$$u=u(a,b)=19a^3-285ab^2-75a^2b+125b^3-1$$

$$v=v(a,b)=5a^3-75ab^2+57a^2b-95b^3$$

Substituting the above values of $u\,\&\,v$ in equation (2), the corresponding integer solutions of (1) are given by

$$x=x(a,b)=24a^3-360ab^2-18a^2b+30b^3-1$$

$$y=y(a,b)=14a^3-210ab^2-132a^2b+220b^3-1$$

$$z=z(a,b)=a^2+5b^2$$

Properties:

1. $y(a,a)-x(a,a)$ is a cubical integer.
2. $y(1,1)-x(1,1)$ is a nasty number.
3. $x(1,a)-y(1,a)+38a\,z(1,a)+50T_{8,a}-26Gno_a\equiv 0 \pmod{36}$
4. $x(b,1)-y(b,1)-5SO_b-19T_{14,b}+25Gno_b\equiv 0 \pmod{215}$
5. $y(a,1)-x(a,1)+4z(a,1)+15O_a+55T_{6,a}-50Gno_a\equiv 0 \pmod{260}$
6. $x(a,a)-y(a,a)+a\,z(a,a)+420P_a^5-21T_{22,a}\equiv 0 \pmod{189}$
7. $y(1,1)-x(1,1)+18z(1,1)$ is a perfect square.

Pattern: 2

Instead of (5), Write $486=(9+9i\sqrt{5})(9-9i\sqrt{5})$ (7)

Substituting (7) and (4) in (3) and employing the method of factorization, following the procedure presented in pattern 1, the corresponding integer solutions of (1) are represented by

$$x = x(a,b) = 18a^3 - 270ab^2 - 108a^2b + 180b^3 - 1$$
$$y = y(a,b) = -162a^2b + 270b^3 - 1$$
$$z = z(a,b) = a^2 + 5b^2$$

Properties:

1. $y(a,a) - x(a,a) - 12a\,z(a,a)$ is a cubical integer.
2. $y(1,1) - x(1,1) - 12z(1,1)$ is a nasty number.
3. $y(a,1) - x(a,1) + 2z(a,1) + 36P_a^5 + 2T_{16,a} + 2T_{22,a} - 120\,Gno_a \equiv 0 \pmod{220}$
4. $x(a,1) - 2a\,z(a,1) - 32P_a^5 + 62T_{6,a} + 171\,Gno_a \equiv 0 \pmod{8}$
5. $x(1,b) - 2z(1,b) - 90\,SO_b + 20T_{30,b} + 139\,Gno_b \equiv 0 \pmod{124}$
6. $x(1,b) - y(1,b) + z(1,b) + 5TO_b + 15O_b + 43T_{22,b} + 104\,Gno_b \equiv 0 \pmod{115}$
7. $y(1,1) - x(1,1) + 6z(1,1)$ is a perfect square.

Pattern: 3

Instead of (5), Write $486 = (21 + 3i\sqrt{5})(21 - 3i\sqrt{5})$ (8)

Substituting (8) and (4) in (3) and following the same procedure presented in pattern 1, the corresponding integer solutions of (1) are represented by

$$x = x(a,b) = 24a^3 - 360ab^2 + 18a^2b - 30b^3 - 1$$
$$y = y(a,b) = 18a^3 - 270ab^2 - 108a^2b + 180b^3 - 1$$
$$z = z(a,b) = a^2 + 5b^2$$

Properties:

1. $y(a,a) - x(a,a) + 8a\,z(a,a)$ is a cubical integer.
2. $y(1,1) - x(1,1) + 8z(1,1)$ is a nasty number.
3. $y(a,1) - x(a,1) + 3a\,SO_a + 63T_{6,a} - 12\,Gno_a \equiv 0 \pmod{222}$
4. $4y(a,1) - 3x(a,1) + 162(T_{8,a} + Gno_a) \equiv 0 \pmod{647}$
5. $x(a,1) - a\,z(a,1) - 46P_a^5 + T_{6,a} + T_{8,a} + 184\,Gno_a \equiv 0 \pmod{215}$
6. $y(a,1) - 2a\,z(a,1) - 24O_a + 12T_{20,a} + 192\,Gno_a \equiv 0 \pmod{13}$
7. $y(1,1) - x(1,1) - 4z(1,1)$ is a perfect square.

Pattern: 4

Instead of (5), Write $486 = \frac{(13 + 29i\sqrt{5})(13 - 29i\sqrt{5})}{9}$ (9)

and using the same procedure as in pattern 1, the corresponding solutions of (3) are represented by

$$u = u(a,b) = \frac{1}{3}(13a^3 - 195ab^2 - 435a^2b + 725b^3) - 1$$
$$v = v(a,b) = \frac{1}{3}(29a^3 - 435ab^2 + 39a^2b - 65b^3)$$
$$z = z(a,b) = a^2 + 5b^2$$

Since our interest is on finding integer solutions, we have choose a and b suitably so that u, v and z are integers. Let us take a = 3A and b = 3B, we have

$$u = u(A,B) = 117A^3 - 1755AB^2 - 3915A^2B + 6525B^3 - 1$$
$$v = v(A,B) = 261A^3 - 3915AB^2 + 315A^2B - 585B^3$$
$$z = z(A,B) = 9A^2 + 45B^2$$

In view of (2), the integer solutions of (1) are given by

$$x = x(A,B) = 378A^3 - 5670AB^2 - 3564A^2B + 5940B^3 - 1$$
$$v = v(A,B) = -144A^3 + 2160AB^2 - 4266A^2B + 7110B^3 - 1$$
$$z = z(A,B) = 9A^2 + 45B^2$$

Properties:

1. $y(A,A) - x(A,A) - 140\,A\,z(A,A)$ is a cubical integer.
2. $y(1,1) - x(1,1)$ is a nasty number.
3. $x(A,1) + y(A,1) - 468P_A^5 + 2688T_{8,A} + 4443Gno_A \equiv 0 \pmod{8605}$
4. $x(A,1) - 42Az(A,1) + 324T_{24,A} + 5400Gno_A \equiv 0 \pmod{539}$
5. $y(A,1) + 16Az(A,1) + 474T_{20,A} + 456Gno_A \equiv 0 \pmod{6653}$
6. $x(A,1) + y(A,1) + 2z(A,1) - 468P_A^5 + 894T_{20,A} + 5331Gno_A \equiv 0 \pmod{7807}$
7. $x(1,1) + y(1,1) - 17z(1,1)$ is a perfect square.

Note:
In addition, one may write 486 as

$$486 = \begin{cases} \dfrac{(153+9i\sqrt{5})(153-9i\sqrt{5})}{49} \\ \dfrac{(117+45i\sqrt{5})(117-45i\sqrt{5})}{49} \\ \dfrac{(113+47i\sqrt{5})(113-47i\sqrt{5})}{49} \\ \dfrac{(87+57i\sqrt{5})(87-57i\sqrt{5})}{49} \\ \dfrac{(37+67i\sqrt{5})(37-67i\sqrt{5})}{49} \\ \dfrac{(3+69i\sqrt{5})(3-69i\sqrt{5})}{49} \end{cases}$$

For these choices, one may obtain different patterns of solutions of (1).

Conclusion:

In this paper, we have presented four different patterns of non-zero distinct integer solutions of the non-homogeneous cone given by $3(x^2+y^2)-4xy+2(x+y+1)=972\,z^3$.

To conclude, one may search for other patterns of non-zero integer distinct solutions and their corresponding properties for other choices of cubic diophantine equations.

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