Static Analysis of Cylindrical Shells

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Abstract - – *In this paper, an isotropic cylindrical shell is studied under the external pressure. For and uniformly distributed load and sinusoidal load at simply supported end conditions, cylindrical shell is studied for its deflection and the von-Mises stresses are analyzed. Analytical modeling is based on first order shear deformation theory (FOST) and a finite element computational tool ABAQUS [2] is used to model the isotropic cylindrical shell.*

Key Words: Isotropic Cylindrical shell,FOST,ABAQUS, von-Mises stresses.

1.INTRODUCTION

Shells are commonly used in many engineering structures, such as aerospace, automotive and submarine structures. Isometric shells are getting more attention recently. Because of the simplicity of shell theories, it is favorable to analyze shell structures with shell theories instead of three dimensional (3D) theories of elasticity. Shell theories redeem the difficulty of shell analyses by employing certain assumptions on the behavior of displacements along the thickness direction. For instance, First order expansion of in-plane displacements leads to First order shear deformation theories (FSDTs).

A difficulty rise and appears in both in strain displacements and stress resultants in the equation of derivation of basic equation of shell. The term was neglected by first analyst of composite thin shells which is understandable for thin shells .Although the importance of the inclusion of the term has been tested for the thicker shells and proves to be essential.

2. Formulation of FSDT

2.1 Introduction

The FSDT developed by Mindlin [12] accounts for the shear deformation effect by the way of a linear variation of the in-plane displacements through the thickness. It is noted that the theory developed by Reissner [13] also accounts for the shear deformation effect. However, the Reissner theory is not similar with the Mindlin theory like erroneous perception of many researchers through the use of misleading descriptions such as “Reissner-Mindlin plates” and “FSDT of Reissner”.The major difference between two theories was established by Wang et al. by derivating the bending relationships between Mindlin and Reissner quantities for a general plate problems. Since the Reissner theory was based on the assumption of a linear bending stress distribution, its formulation will inevitably lead to the displacement variation being not necessarily linear across the plate thickness. Thus, it is incorrect to refer to the Reissner theory as the FSDT which implies a linear variation of the displacements through the thickness. Another difference between two theories is that the normal stress which was included in the Reissner theory was omitted in the Mindlin.

The FSDT was used to model FG shells. Reddy and Chin [14] studied the dynamic response of FG cylinders and plates subjected to two different types of thermal loadings using the FSDT and the finite element method.

2.2 Definition of Displacement field

$$u = u_0 + z\theta_x$$

(1a)

$$v = v_0 + z\theta_y$$

$$w = w_0$$

In the above relations, the terms u, v and w are the displacements of a general point (x, y, z) in the domain in *x, y* and *z* directions respectively. The parameters u_0, v_0 are the inplane displacements and w_0 is the transverse displacement of a point (*x, y*) on the element middle plane. The functions θ_x and θ_y are the rotations of the normal to the element middle plane about *y*- and *x*-axis respectively.

2.3 Strain displacement relation

With the definition of strains from the linear theory of elasticity, assuming $h/R_x, h/R_y \ll 1$, the general strain-displacement relations in the curvilinear co-ordinate system are given as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{w}{R_x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R_y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \frac{u}{R_x}$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \frac{v}{R_y}$$

Substituting the expressions for displacements at any point within the space given by Eqs. (1a) for the displacement models considered herein, the linear strains in terms of middle surface displacements, for each displacement model can be obtained as follows:

$$\varepsilon_x = \varepsilon_{x0} + z\chi_x$$
$$\varepsilon_y = \varepsilon_{y0} + z\chi_y$$
$$\varepsilon_z = 0$$
$$\gamma_{xy} = \varepsilon_{xy0} + z\chi_{xy}$$
$$\gamma_{xz} = \emptyset_x + z\chi_{xz}$$
$$\gamma_{yz} = \emptyset_y + z\chi_{yz}$$

Where, $(\chi_{xz}, \chi_{yz}) = (-\frac{\theta_x}{R_x}, -\frac{\theta_y}{R_y})$

2.4 Stress-strain relations and stress resultants

Assuming the principal material axes (1,2,3) and the shell axes (x,y,z) in the curvilinear co-ordinate system ,the three dimensional stress-strain relations for an cylindrical shell with reference to the principal material axes for the theory to be developed based on the displacement are defined as follows:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix}$$

(2a)

Where,

$$C_{11} = \frac{E_1}{(1-\nu_{12}\nu_{21})}, \quad C_{12} = \frac{\nu_{21}E_1}{(1-\nu_{12}\nu_{21})},$$
$$C_{22} = \frac{E_2}{(1-\nu_{12}\nu_{21})}, \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2},$$
$$C_{33} = G_{12}, \quad C_{44} = G_{13}, \quad C_{55} = G_{23}$$

These equations in compacted form may be written as

$$\boldsymbol{\sigma} = \boldsymbol{C\varepsilon}$$

As mentioned earlier, the relations given by Eq. (2a) are the stress-strain constitutive relations for the cylindrical shell referred to shell's principal material axes (1,2,3). The principal material axes of shell may not coincide with the reference axes of the shell (x,y,z).It is therefore necessary to transform the constitutive relations from the shell's material axes (1,2,3) to reference axes (x,y,z).This is conveniently accomplished through the transformations .The final relations are as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

(2b)

These equations in compacted form may be written as

$$\boldsymbol{\sigma} = \boldsymbol{Q\,\varepsilon}$$

in which the coefficients of the Q matrix, called as reduced elastic constants are defined in Appendix A.

The components of the strain vector $\bar{\varepsilon}$ and the corresponding components of the stress- resultant vector $\bar{\sigma}$ are defined as follows:

$$\begin{bmatrix} N_x & M_x \\ N_y & M_y \\ N_{xy} & M_{xy} \end{bmatrix} = \sum_{L=1}^{NL} \int_{Z_L}^{Z_{L+1}} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} (1, z^2, z, z^3)dz$$
$$= \sum_{L=1}^{NL} \int_{Z_L}^{Z_{L+1}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{12} & Q_{22} & Q_{23} & Q_{24} \\ Q_{13} & Q_{23} & Q_{33} & Q_{34} \\ Q_{14} & Q_{24} & Q_{34} & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} (1, z^2, z, z^3)dz$$

$$\begin{bmatrix} Q_x & S_x \\ Q_y & S_y \end{bmatrix} = \sum_{L=1}^{NL} \int_{Z_L}^{Z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} (1, z^2, z, z^3)dz$$

$$\bar{\sigma} = (N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, Q_x, Q_y, S_x, S_y)$$
$$\bar{\varepsilon} = (\varepsilon_{x0}, \varepsilon_{y0}, \varepsilon_{xy0}, \chi_x, \chi_y, \chi_{xy}, \emptyset_x, \emptyset_y, \chi_{xz}, \chi_{yz})$$

2.5 Equilibrium equations

For equilibrium equations, the total potential energy must be stationary and using the definitions of stress-resultants and mid-surface strains stated in above sections principal of virtual work yields

$$\delta\Pi = \delta(U - W) = 0$$

Where, U is the strain energy and W represents the work done by external forces. These are evaluated as follows:

$$\delta U = \int_x \int_y \int_z (\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \sigma_z \delta \varepsilon_z + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz})dx\,dy\,dz. \quad (3a)$$

Integration through thickness and by substituting in terms of strains and introducing stress resultants, the above relations transform in the following form

$$\delta \Pi = \int_x \int_y (N_x \delta \varepsilon_{x0} + N_y \delta \varepsilon_{y0} + N_{xy} \delta \varepsilon_{xy0} + M_x \delta \chi_x + M_y \delta \chi_y + M_{xy} \delta \chi_{xy} + Q_x \delta \emptyset_x + Q_y \delta \emptyset_y + S_x \delta \chi_{xz} + S_y \delta \chi_{yz} - \delta w_0 q)\,dxdy = 0 \quad (3b)$$

In the above equation is the distributed transverse load. The governing equations of equilibrium can be derived from eq. (3b) by integrating the displacement gradients in mid surface strains by parts and setting the coefficients of derivatives of mid-surface displacements to zero separately. Thus one obtains the following equilibrium equations.

$$\frac{\partial N_x}{\partial x} + \frac{\partial (N_{xy} + c_0 M_{xy})}{\partial y} + \frac{Q_x}{R_x} = 0$$
$$\frac{\partial N_y}{\partial y} + \frac{\partial (N_{xy} - c_0 M_{xy})}{\partial x} + \frac{Q_y}{R_y} = 0$$
$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \frac{N_x}{R_x} - \frac{N_y}{R_y} + q = 0$$
(3c)
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + \frac{S_x}{R_x} = 0$$
$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + \frac{S_y}{R_y} = 0$$

In addition following line integrals are also obtained

$$\int_x (N_y \delta v_0 + N_{xy} \delta u_0 + M_y \delta \theta_y + M_{xy} \delta \theta_x + C_0 M_{xy} \delta u_0 + Q_y \delta w_0 + S_y \delta \theta_z)dx + \int_y (N_x \delta u_0 + N_{xy} \delta v_0 + M_x \delta \theta_x + M_{xy} \delta \theta_y - C_0 M_{xy} \delta v_0 + Q_x \delta w_0 + S_x \delta \theta_z) \quad (3d)$$

2.6 Closed form solutions

Sinusoidal variation of transverse load is considered as under:

$$q = \sum_{m,n}^{\infty} q_{mn} \sin \alpha x \sin \beta y$$

$\alpha = \frac{m\pi}{a}, \beta = \frac{m\pi}{b}$ in which a and b are the dimensions of shell middle surface along the x and y axes respectively.

The exact form of spatial variation of mid-surface displacements is given by

$$u_0 = \sum_{m,n}^{\infty} u_{0mn} \cos \alpha x \sin \beta y$$
$$v_0 = \sum_{m,n}^{\infty} v_{0mn} \sin \alpha x \cos \beta y$$
$$w_0 = \sum_{m,n}^{\infty} w_{0mn} \sin \alpha x \sin \beta y$$
(4a)
$$\theta_x = \sum_{m,n}^{\infty} \theta_{xmn} \cos \alpha x \sin \beta y$$
$$\theta_y = \sum_{m,n}^{\infty} \theta_{ymn} \sin \alpha x \cos \beta y$$

3. Analysis in ABAQUS

- To find the dimensionless displacements of an isotropic cylindrical shells having following properties.

  a/b= 1, $k^2 = 5/6$, υ=0.25

1. Find displacement for a/h=20 and for different ratios of
   a) a/R= 0.5
   b) a/R=1
   c) a/R=2
2. Find the displacement for a/h=10, and for different ratios of
   a) a/R= 0.5
   b) a/R=1
   c) a/R=2

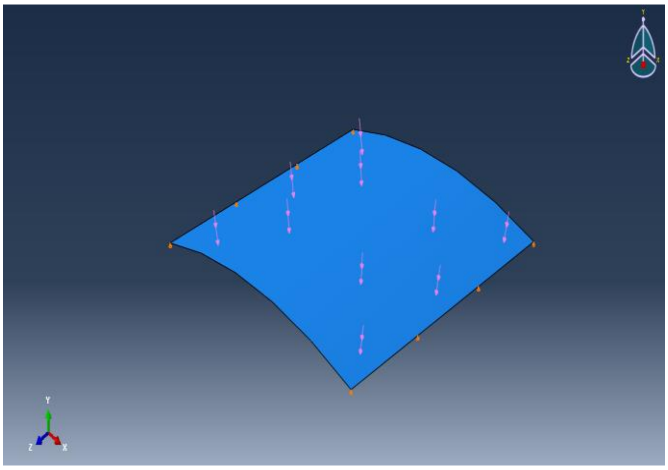

Fig-1 Modeling in ABAQUS

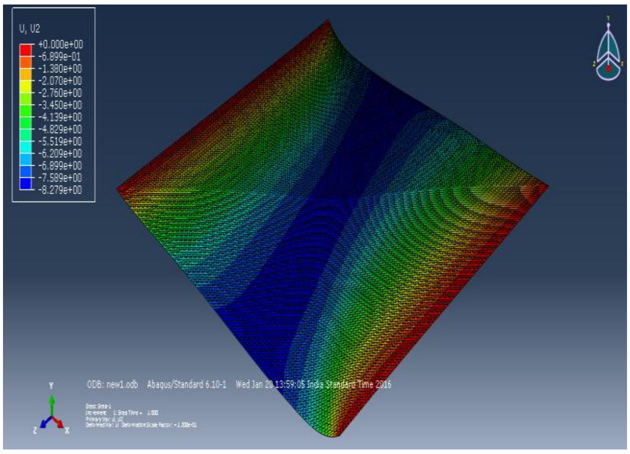

Fig-2 Shows stresses and deflection

Table 1. Non-Dimensional transverse deflection for longitudinal edges simply supported cylindrical shell under uniformly distributed loading with a/h = 20.
[ ] % error w.r.t. FSDT results

a/R	3D	FSDTQ	FSDT	ABAQUS (present)
0.5	26.698	26.679	26.596	25.87 [2.72%]
1	11.427	11.388	11.337	11.19 [1.29%]
2	1.6384	3.0070	2.9994	2.92 [2.63%]

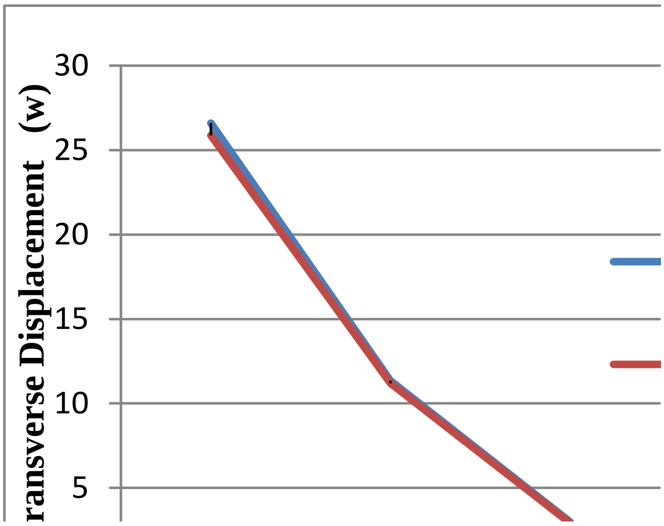

Fig.3. Non- dimensional transverse displacement vs. FSDT results having a/h = 20.

Table 2. Non-Dimensional transverse deflection for longitudinal edges simply supported cylindrical shell under uniformly distributed loading with a/h = 10.

a/R	3D	FSDTQ	FSDT	ABAQUS (present)
0.5	40.875	40.956	40.699	41.1 [0.1%]
1	28.415	28.333	28.009	29 [3.53%]
2	12.242	12.108	11.972	11.52 [3.77%]

[ ] % error w.r.t. FSDT results

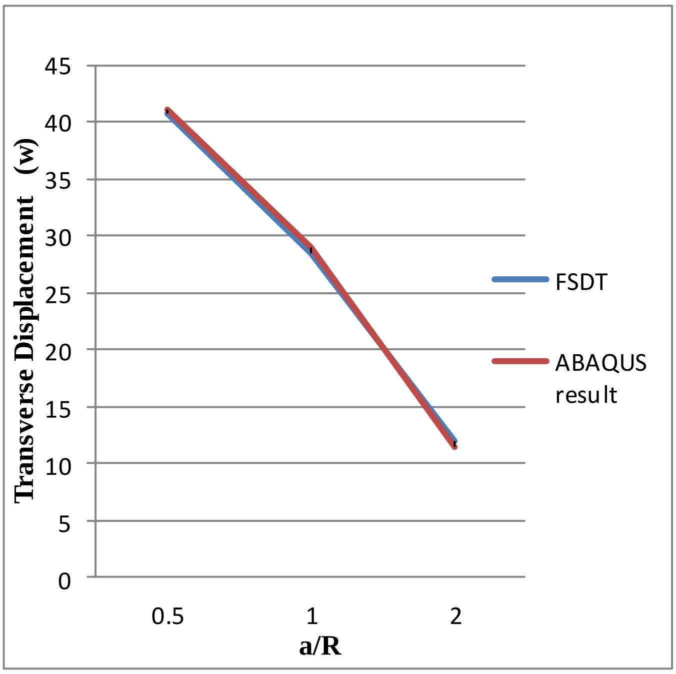

Fig.4. Non- dimensional transverse displacement vs. FSDT results having a/h = 10.

3. CONCLUSIONS

It is concluded that for a simply supported isotropic cylindrical shell subjected to uniformly distributed load, for a constant ratio of a/central displacement decreases as a/R ratio increases.

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