On the Exponential Diophantine Equation $36^x + 3^y = z^2$

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Abstract: *In this paper we search for unique non negative integer solutions of the Diophantine equation*

$$36^x + 3^y = z^2$$

The solution (x,y,z) are (0,1,2) and (2,6,45).The second set of solution implies that (4,6,45) is the solution (x,y,z) of the exponential Diophantine equation $6^x + 3^y = z^2$ *where x,y and z are non negative integers.*

Keywords: exponential Diophantine equation, integer solution.

Mathematical classification: AMS Mathematical subject classification (2010): 11D61

Introduction:

A linear Diophantine equation is an equation that sums two monomials of degree zero or one. An exponential Diophantine equation is one in which exponents on terms can be unknowns. A general theory of exponential Diophantine equations is not available. Majority of the equations are solved via adhoc methods such as Catalan's conjecture or even trial and error method.

For various problems and ideas one may refer [1] and [2].[3] to [10] has been studied for various methods of solving exponential Diophantine equations.

In this paper we search for unique non negative integer solutions of the exponential Diophantine equation $36^x + 3^y = z^2$

Preliminaries:

Catalan's conjecture:

The Diophantine equation $a^x + b^y = 1$ has unique integer solution with min {a, b, x, y} > 1.The solution (a, b, x, y) is (3, 2, 2, 3).This was proved by Mihailescu in 2004.

Method of Analysis:

In this section we prove that $36^x + 3^y = z^2$ has non negative integer solution. The solution (x,y,z) is (0,1,2) and (2,6,45). The second set of solution implies that (4,6,45) is the solution (x,y,z) of the exponential Diophantine equation $6^x + 3^y = z^2$ where x, y and z are non-negative integers.

Theorem: (0, 1, 2) and (2, 6, 45) are solutions of (x, y, z) of the Diophantine equation $36^x + 3^y = z^2$ where x, y and z are non-negative integers.

Proof:

We will divide the proof under three cases on y.

Case (i): Take $x \neq 0$

Suppose y = 0

Then $36^x + 1 = z^2$

$6^{2x} = (z+1)(z-1)$

Let $(z-1) = 6^u$ (1)

where u is non-negative integer

Implies $6^{2x} = (z+1)6^u$

$\Rightarrow \quad (z+1) = 6^{2x-u}$ (2)

(2) - (1) $\Rightarrow 6^u[6^{2x-2u} - 1] = 2$

$\Rightarrow$ u = 0

$\Rightarrow 6^{2x} = 3$

This is impossible for positive values of x

Therefore $y \neq 0$

Case (ii): $x = 0$

Write $36^x + 3^y = z^2$ as $6^{2x} + 3^y = z^2$

implies $3^y = (z-6^x)(z+6^x)$

Let $z - 6^x = 3^w$ (3)
Where w is non-negative integer.

$\Rightarrow z + 6^x = 3^{y-w}$ (4)

(4) - (3) $\Rightarrow$ $3^w(3^{y-2w} - 1) = 2(6^x)$ (5)
$\Rightarrow$ w = x

If x = 0 then w = 0 and $3^{y-2w} - 1 = 2$
$\Rightarrow$ y = 1.
In this case z = 2
Hence (x, y, z) = (0, 1, 2)

Case (iii): $x \geq 2$

Proceeding as in case (ii) from (5) we have

$3^w(3^{y-2w} - 1) = 2(6^x)$
If x = 2 then w = 2, implies y = 6
In this case z = 45
Hence (x, y, z) = (2, 6, 45)

Corollary:

(4, 6, 45) is a solution of (x, y, z) of the exponential Diophantine equation $6^x + 3^y = z^2$, where x, y and z are non-negative integers.

Proof:

By the theorem we have $36^2 + 3^6 = 45^2$

$\Rightarrow 6^4 + 3^6 = 45^2$
Therefore (4, 6, 45) is a solution of the exponential Diophantine equation $6^x + 3^y = z^2$ where x, y and z are non-negative integers.

Conclusion:

In this paper we have found (0, 1, 2) and (2, 6, 45) are exact solutions to $36^x + 3^y = z^2$ in non negative integers. One may also search for other set of solution for similar type of equations.

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