A Review article on some generalizations of Banach's contraction principle

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Abstract:

Let (X,d) be a metric space. The well known Banach's Contraction Principle states that if T: X $\rightarrow$ X is a contraction on X(i.e.d (Tx,Ty) $\leq$ c d (x,y) for some $0 \leq c<1$ and for all x,y in X) and (X,d) is complete then T has a fixed point in X (i.e. Tx = x for some x in X) In this paper , a number of extensions of Banach contraction principle have been discussed .

Keywords:

Contraction , Multivalued contractions, quasi contraction, λ - generalized contraction , f-contraction and generalized f-contraction, ϕ -weak contraction, ω - distance, weakly uniform strict contraction

Introduction:

S.Nadler has extended the result to multivalued contractions:

Let (X,d) be a complete metric space

(a)CB(X)={C : C is closed and bounded subset of X}

(b)N(ε,C)={x$\in$X:d (x,c)< ε for some c$\in$C}

(c)H(A,B)=inf{ ε: A$\subset$N(ε,B) or B$\subset$N(ε,A)} where A , B$\in$CB(X)

The function H is a metric on CB(X) and is called Hausdorff metric.

Let (X,d_1) and (Y,d_2) be two metric spaces.

Definition : A function T: X$\rightarrow$CB(Y) is called multivalued contraction mapping of X into Y iff H(Tx,Ty)$\leq$ cd_1(x,y) for some 0<c<1 and for all x,y in X

A point x is said to be fixed point of T if x $\in$ Tx

Therom : Let (X,d) be a complete metric space . If T : X$\rightarrow$CB(X) is a multivalued contraction mapping then T has a fixed point (i.e. there exists x$\in$X such that x $\in$ Tx).

Above theorem generalizes the Banach's Contraction Principle.(the map J: X $\rightarrow$CB(X) defined by J(x)={x} is an isometry. T: X$\rightarrow$X be a contraction then J$\circ$T: X$\rightarrow$CB(X) is a multivalued contraction,thus there exists x$\in$X such that x$\in$J$\circ$T(x) i.e. x $\in$J(T(x)) which implies x$\in${Tx}.Thus x=Tx

Lj.B.Ciric and S.B. Presic extended the result as follows:

Theorem : Let (X,d) be a complete metric space , k a positive integer and T: $X^K \to X$ a map satisfying the condition:

$d(T(x_1,x_2,\ldots x_k),T(x_2,x_3,\ldots x_{k+1})) \leq c\ \max\{d(x_i,x_{i+1}) : 1 \leq i \leq k\}$

where $c \in [0,1)$ is a constant and $x_1,x_2,\ldots x_{k+1}$ in X are arbitrary then there exists a point x in X such that T(x, x, . . . x) = x
If in addition , on diagonal $\Delta \subset X^K$, d(T(u , u ,...u),T(v ,v ,...v)) < d(u,v) holds for all u,v in X with u $\neq$ v then x is the unique point in X with T(x ,x...x) = x

For k=1 the above theorem gives Banach's contraction principle.

James Merryfield and James D. Stein gave the proof of generalized banach contraction conjecture :

Theorem : Let (X,d) be a complete metric space.T : $X \to X$ be a map and let $0 \leq c < 1$.Let J be a positive integer .Assume for each pair x , y in X

$\text{Min}\{d(T^K x, T^K y) : 1 \leq k \leq J\} \leq c\ d(x,y)$ then T has a unique fixed point.

For J = 1, the above theorem gives Banach's contraction principle.

Lj.B.Ciric has extended the result to quasi contractions:

Definition : A mapping T : $X \to X$ is said to be quasi-contraction if there exists a number $0 \leq c < 1$ such that

$d(Tx,Ty) \leq c\ \max\{d(x.y), d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\}$ for all x , y in X

For $x \in X$, Let $O(x,\infty)=\{x, Tx, T^2x, \ldots\}$

Definition : A space X said to be T-orbitally complete if every Cauchy sequence in $O(x,\infty)$ for some $x \in X$ is convergent in X.

Theorem : Let T : $X \to X$ be a quasi - contraction and X is T – orbitally complete.Then T has a unique fixed point in X.

Above theorem is clearly generalization of Banach's Contraction Principle as every contraction is quasi contraction.

M.S. Khan introduced altering distance function to generalize the result:

Definition : A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following conditions are satisfied:

(a) $\psi(0)=0$ (b) ψ is continuous and monotonically non-decreasing.

Theorem : Let (X,d) be a complete metric space.Let ψ be an altering distance function and letT: $X \to X$ be a map which satisfies the following inequality:

$\psi(d(Tx,Ty)) \leq c\ \psi\, d(x,y)$ for all x,y in X and for some $0 \leq c < 1$ then T has a unique fixed point.

Clearly for $\psi(t)=t$, above theorem gives Banach's Contraction Principle.

Lj.B. Ciric introduced λ – generalized contraction to extend the result:

Definition : A mapping T: X $\rightarrow$ X is said to be a λ – generalized contraction if for every x,y in X, there exist non negative numbers q(x,y), r(x,y), s(x,y), t(x,y) such that $\sup_{x,y \in X}$ {q(x,y)+r(x,y)+s(x,y)+2t(x,y)}= λ <1 and d(Tx,Ty) $\leq$ q(x,y) d(x,y)+r(x,y) d(x,Tx)+s(x,y) d(y,Ty)+t(x,y) (d(x.Ty)+d(y,Tx)) holds for all x,y in X

Theorem : Let T be λ - generalized contraction of T –orbitally complete metric space X into itself then T has a unique fixed point in X.

If we take r(x,y)=s(x,y)=t(x,y)=0 and q(x,y)=c for all x,y where 0<c<1 then above theorem gives Banach contraction principle.

Milan R.Tascovic

Theorem : Let T : X $\rightarrow$ X be a map and X be T – orbitally complete .If T satisfies the following condition:

There exist real numbers α_i , β for every x,y in X such that : $\alpha_1+\alpha_2+\alpha_3>\beta$ and $\beta-\alpha_2\geq 0 \vee \beta-\alpha_3\geq 0$

and α_1 d(Tx ,Ty)+ α_2 d(x,Tx)+ α_3 d(y,Ty)+ α_4 min{d(x,Ty),d(y,Tx)} $\leq$ β d(x,y) then T has a fixed point.

Above theorem generalizes banach's contraction principle as every contraction satisfies the above condition.

Milan R.Tascovic also defined f-contraction and generalized f-contraction :

Definition : A mapping f: $\mathbf{R}_+^k \rightarrow \mathbf{R}_+$ is semihomogenous if f($\delta x_1, \delta x_2, \ldots \delta x_k$) $\leq$ δ f($x_1, x_2, \ldots x_k$): δ >0

A mapping T : X $\rightarrow$ X is said to be f- contraction if for every x,y in X there exist non negative real numbers α_i (x,y) i=1,2,...5 such that

d(Tx,Ty) $\leq$ f(α_1 (x,y)d(x,y), α_2 (x,y)d(x,Tx), α_3 (x,y)d(y,Ty), α_4 (x,y)d(x.Ty), α_5 (x,y)d(y,Tx))

where sup{f($\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$): x , y $\in$ X}= λ <1and the existing mapping f:$(\mathbf{R}_+^0)^5 \rightarrow \mathbf{R}_+^0$ is increasing and semihomogenous .

Theorem : Let T be a f-contraction on a metric space X and let X be T orbitally complete then T has a unique fixed point.

Every contraction mapping satisfies the above condition for f(s,t,u,v,w)= s and α_1 = c, $\alpha_2=\alpha_3=\alpha_4=\alpha_5$ =0 where 0<c<1 ,thus above theorem generalizes Banach's contraction principle.

Definition : A mapping f: $\mathbf{R}_+^K \rightarrow \mathbf{R}_+^K$ is semihomogenous of order $\alpha \geq 1$ iff f($\delta x_1, \delta x_2, \ldots \delta x_k$) $\leq$ δ f($x_1, x_2, \ldots x_k$)

Where δ belongs to $[\alpha, \infty)$

Definition : A mapping T : X $\rightarrow$ X is said to be generalized f- contraction if for every x,y in X

d(Tx,Ty) $\leq$ f(d(x,y), d(x,Tx), d(y,Ty), d(x.Ty), d(y,Tx))

where f : $(\mathbf{R}_+^0)^5 \rightarrow \mathbf{R}_+^0$ is increasing,semihomogenous of order $\alpha \geq 1$ and with the properties f(t,t,....t) < t $\wedge$

$\lim_{y \to t+0} \sup f(y,y,\ldots y) < t$ for all $t \in (0, \alpha]$

Theorem : Let T be a genertalized f- contraction on a metric space X and let X be T orbitally complete then T has a unique fixed point.

Every contraction mapping satisfies the above condition for f(s,t,u,v,w)= c s , where 0<c<1 ,thus above theorem generalizes Banach's contraction principle.

A.Meir and Emmett Keeler gives an $\in$ - δ condition to generalize the result:

Theorem : Let (X,d) be a complete metric space . T : X $\rightarrow$ X be a map satisfying the following weakly uniform strict contraction :

Given $\in > 0$ there exists $\delta > 0$ such that

$\in \leq d(x,y) < \in + \delta \Rightarrow d(Tx, Ty) < \in$

Above theorem generalizes the Banach's contraction principle since every contractive mapping is uniformly continuous and every uniformly continuous map satisfies the above condition.

Chi Song Wong :

Theorem : Let S and T be self mappings of a complete metric space X. Suppose there exist functions α_i ,i=1,2, ...5 from $X \times X$ into $[0, \infty)$ such that

(a) $r = \sup \{ \sum_{i=1}^{5} \alpha_i(x,y) : x,y \in X\} < 1$
(b) $\alpha_2 = \alpha_3, \alpha_4 = \alpha_5$
(c) For any distinct point x, y in X $d(Sx, Ty) \leq \alpha_1(x,y)\, d(x,y) + \alpha_2(x,y)\, d(x,Ty) + \alpha_3(x,y)\, d(y,Sx) + \alpha_4(x,y)\, d(x,Sx) + \alpha_5(x,y)\, d(y,Ty)$

then S or T has a fixed point .If both S and T has fixed points then each of S and T has a unique fixed point and these two fixed ponts coincide

every contraction mapping T satisfies the above condition with S=T, $\alpha_1(x,y) = c$; 0<c<1 , $\alpha_2(x,y) = \alpha_3(x,y) = \alpha_4(x,y) = \alpha_5(x,y) = 0$, thus above theorem generalizes the Banach's contraction principle.

Theorem : T be a self mapping of a complete metric space (X,d).Suppose that there exist functions

α_i , i=1,2,3,4,5 of $(0, \infty)$ into $[0, \infty)$ such that

Each α_i is upper semicontinuous from the right such that $\alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t) + \alpha_5(t) < t$, $t > 0$ and for any distint points x,y in X ,

$d(Tx,Ty) \leq a_1 d(x,Tx)+a_2 d(y,Ty)+a_3 d(x,Ty)+a_4 d(y,Tx)+a_5 d(x,y)$ where $a_i = \alpha_i (d(x,y))/d(x,y)$

Then T has a unique fixed point.

Every contraction mapping satisfies the above condition for α_5(t) = c t where 0<c<1 and $\alpha_1(t)=\alpha_2(t)=\alpha_3(t)=\alpha_4(t)=0$ for all t . Thus above theorem generalizes Banach's contraction principle.

B.E. Rhoades made use of ϕ –weak contraction to generalize the result:

Definition : let (X,d) be a metric space.T: $X \rightarrow X$ be a map.T is said to be ϕ –weak contraction if

$d(Tx,Ty) \leq d(x,y)- \phi(d(x,y))$ where $\phi:[0,\infty) \rightarrow [0,\infty)$ is continuous and non decreasing function with

$\phi(t)=0$ iff t=0

Theorem : If (X,d) is complete metric space and T is a ϕ -weak contraction on X then T has a unique fixed point.

Taking $\phi(t)=(1-c)t$,where 0<c<1 ,the above theorem gives Banach's contraction principle.

P.N. Dutta and B.S. Chaudhury:

Theorem : Let (X,d) be a complete metric space and let T: $X \rightarrow X$ be a map satisfying

$\psi(d(Tx,Ty)) \leq \psi(d(x,y))-\phi(d(x,y))$ for all x,y in X where $\phi,\psi: [0,\infty) \rightarrow [0,\infty)$ are both continuous and monotonic decreasing with $\psi(t)=\phi(t)=0$ iff t=0.Then T has a unique fixed point.

Taking $\psi(t)=t$ and $\phi(t)=(1-c)t$ where 0<c<1 ,above theorem gives Banach's contraction principle.

Mark Voorneveld :

Definition : Let X be a metric space with metric d. We define an ω - distance on X to be a function $\rho : X \times X \rightarrow [0,\infty)$ such that

(a) ρ satisfies the triangle inequality

(b) $\rho(x.) : X \rightarrow [0,\infty)$ is lower semi continuous for every $x \in X$ i.e. if $y_m \rightarrow y$ then

$\rho(x,y) \leq \liminf_{m\to\infty} \rho(x,y_m)$

(c) for every $\in > 0$, there exists a $\delta > 0$ such that for every x,y,z in X if $\rho(z,x) \leq \delta$ and $\rho(z,y) \leq \delta$ then $\rho(x,y) \leq \in$

Definition : Let (X,d) be a metric space and ρ an ω - distance on X. Let F(ρ) denote the family of functions α on $X \times X$ satisfying the following conditions:

(a) For each (x,y) in $X \times X$, $\alpha(x,y)$ depends only on the ω - distance $\rho(x,y)$, this allows us to write $\alpha(\rho(x,y))$ instead of $\alpha(x,y)$
(b) $0 \leq \alpha(d) < 1$ for every d > 0

(c) α (d) is increasing function of d.

Theorem : Let (X,d) be a complete metric space , ρ an ω - distance on X and T :X $\rightarrow$ X a map . If there exists $\alpha \in F(\rho)$ such that ρ (Tx,Ty) $\leq$ α (x,y) ρ (x,y) then T has a unique fixed point x in X.

Above theorem gives banach's contraction principle for ρ = d and α a constant in [0,1).

Maher Berzig :

Definition : let $\psi, \phi : [0, \infty) \rightarrow \mathbf{R}$ be two functions . The pair of functions (ψ, ϕ) is said to be a pair of shifting distance function if the following conditions hold:

a) For u ,v $\in [0, \infty)$ if ψ (u) $\leq$ ϕ (v) then u $\leq$ v
b) For $\{u_n\},\{v_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = w$ if $\psi(u_n) \leq \phi(v_n)$ for all n $\geq$ 0 then w = 0

Theorem : Let (X,d) be a complete metric space . T: X $\rightarrow$ X be a mapping .Suppose there exists a pair of shifting distance functions (ψ, ϕ) such that ψ (d(Tx,Ty)) $\leq$ ϕ (d(x,y)) for all x , y in X then T has a unique fixed point in X .

Above theorem generalizes banach's contraction principle since for a contraction mapping , above conditions are satisfied with ψ (x) = x and ϕ (x) = c x where 0 < c <1

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