A Study on the Root Systems and Dynkin diagrams associated with $QHA_2^{(1)}$

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Abstract - *In this paper, a specific class of indefinite quasi-hyperbolic Kac-Moody algebra $QHA_2^{(1)}$ is considered. We obtain the complete classification of the Dynkin diagrams associated with the quasi hyperbolic indefinite type of Kac-Moody algebra $QHA_2^{(1)}$. The properties of strictly imaginary and purely imaginary roots are also studied for the quasi hyperbolic type Kac-Moody algebra $QHA_2^{(1)}$*

Key Words: Kac-Moody algebras, realization, finite, affine, indefinite, strictly imaginary and purely imaginary roots.

1.INTRODUCTION

A rapidly growing subject of modern mathematics, the Kac-Moody algebras associated with the Generalized Cartan Matrix was introduced and developed by Kac and Moody simultaneously and independently in 1968. The theory of Kac-Moody algebras has significant applications in various branches of mathematics such as Combinatorics, Number theory, Non-linear differential equations and Mathematical physics. The Kac-Moody algebras are divided into finite, affine and indefinite types. In recent years, the indefinite type of Kac-Moody algebras is growing exponentially.

The roots of Kac-Moody algebra can be divided into the real and imaginary roots ([3], [11]). The concepts of strictly imaginary roots for Kac-Moody algebras were introduced by Kac in [4]. The complete classification of Kac-Moody algebras possessing strictly imaginary property was studied by Casperson [2]. In [12], the notion purely imaginary roots Kac-Moody algebras was introduced and the complete classification of Kac-Moody algebras possessing purely imaginary property were also obtained by Sthanumoorthy and Uma Maheswari.

In [5,6 and 8], Kang has computed the structure and the root multiplicities for roots upto level 5 for $HA_1^{(1)}$ and $HA_2^{(2)}$ upto level 3, using homological and spectral sequences theory as in [1]. Root multiplicities of the indefinite type of Kac-Moody Lie algebra $HA_n^{(1)}$ were also obtained in [7]. A new class of Extended-hyperbolic Kac-Moody algebras was introduced in [10] and also the root multiplicities of roots for a particular class of extended-hyperbolic Kac-Moody algebra $EHA_1^{(1)}$ were obtained by Sthanumoorthy and Uma Maheswari. In [13-15], some more general classes of $EHA_1^{(1)}$ and $EHA_2^{(2)}$ were considered and the root multiplicities for roots upto level 3 were also determined.

Another new class of indefinite, non-hyperbolic type of Kac-Moody algebra called quasi-hyperbolic algebra was introduced by Uma Maheswari in [16]. For the particular classes of indefinite, non-hyperbolic type of Kac-Moody algebras QHG_2, $QHA_2^{(1)}$, $QHA_4^{(2)}$, $QHA_5^{(2)}$ and $QHA_7^{(2)}$ were considered and are realized as graded Kac-Moody algebras of quasi hyperbolic type in [17 - 21]. For the same, using the homological theory and spectral sequences techniques developed by Benkart et al. [1] and Kang [5-8], we determined the homology modules upto level three and the structure of the components of the maximal ideals upto level four. In [22], the complete classification of the Dynkin diagrams and some properties of real and imaginary roots for the associated Quasi affine Kac Moody algebras $QAC_2^{(1)}$ was obtained.

In this work, we are going to consider the particular class of quasi-hyperbolic type of Kac-Moody algbra $QHA_2^{(1)}$ whose associated GCM is $\begin{pmatrix} 2 & -1 & -1 & -a \\ -1 & 2 & -1 & -b \\ -1 & -1 & 2 & -c \\ -p & -q & -r & 2 \end{pmatrix}$ where a, b, c, p, q, r are non-negative integers. A complete classification of the Dynkin diagrams associated with the quasi hyperbolic indefinite type of Kac-Moody algebra $QHA_2^{(1)}$ is given. The properties of strictly imaginary roots and purely imaginary roots are also studied for $QHA_2^{(1)}$.

1.1 Preliminaries

In this section, we recall some fundamental concepts regarding Kac-Moody algebras, strictly imaginary roots and purely imaginary roots ([4], [2] and [12]).

Definition 1.1[3]: An integer matrix $A=(a_{ij})_{i,j=1}^{n}$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:

(C1) $a_{ii}=2 \ \forall \ i=1,2,\ldots,n$

(C2) $a_{ij}=0 \Leftrightarrow a_{ji}=0 \ \forall \ i,j=1,2,\ldots,n$

(C3) $a_{ij}\le 0 \ \forall \ i,j=1,2,\ldots,n$

Let us denote the index set of A by N = {1,…,n}. A GCM A is said to decomposable if there exist two non-empty subsets I, J ⊂ N such that I ∪ J = N and $a_{ij} = a_{ji} = 0$ ∀ i ∈ I and j ∈ J. If A is not decomposable, it is said to be indecomposable.

Definition 1.2[3]: A GCM A is called symmetrizable if A = DB, where B is symmetric and D is a diagonal matrix such that D = diag(q_1,…,q_n), with $q_i > 0$ and q_i's are rational numbers.

Definition 1.3[3]: A realization of a matrix $A=(a_{ij})_{i,j=1}^{n}$ is a triple (H, π, π^v), where *l* is the rank of A, H is a 2n - *l* dimensional complex vector space, π = {α_1,…,α_n} and π^v = { α_1^v,…,α_n^v} are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^v) = a_{ij}$ for i, j = 1,….,n. π is called the root basis. Elements of π are called simple roots. The root lattice generated by π is $Q=\sum_{i=1}^{n} Z\alpha_i$.

Definition 1.4[3]: The Kac-Moody algebra g(A) associated with a GCM $A=(a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i , f_i , i= 1,2,..,n and H with the following defining relations :

$$[h,h']=0, \quad h,h'\in H$$
$$[e_i,f_j]=\delta_{ij}\alpha_i^v$$
$$[h,e_j]=\alpha_j(h)e_j$$
$$[h,f_j]=-\alpha_j(h)f_j, \ i,j\in N$$
$$(ade_i)^{1-a_{ij}}e_j=0$$
$$(adf_i)^{1-a_{ij}}f_j=0, \ \forall\ i\neq j,\ i,j\in N$$

The Kac-Moody algebra g(A) has the root space decomposition $g(A)=\underset{\alpha\in Q}{\oplus} g_\alpha(A)$ where $g_\alpha(A)=\{x\in g(A)/[h,x]=\alpha(h)x, \ for\ all\ h\in H\}$. An element α, α ≠ 0 in Q is called a root if $g_\alpha \neq 0$. Let $Q=\sum_{i=1}^{n} Z_+\alpha_i$. *Q* has a partial ordering "≤" defined by α ≤ β if β-α ∈ Q_+, where α , β ∈ *Q*.

Definition 1.5[3]: For any α ∈ *Q* and $\alpha=\sum_{i=1}^{n}k_i\alpha_i$ define support of α, written as supp α , by supp $\alpha=\{i\in N/k_i\neq 0\}$.

Let Δ (=Δ(A)) denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Proposition 1.6[2]: A GCM $A=(a_{ij})_{i,j=1}^{n}$ is symmetrizable if and only if there exists an invariant, bilinear, symmetric, non degenerate form on g(A).

Definition 1.7[3]: To every GCM A is associated a Dynkin diagram S(A) defined as follows: S(A) has n vertices and vertices i and j are connected by max {|a_{ij}|,|a_{ji}|} number of lines if $a_{ij} . a_{ji} \le 4$ and there is an arrow pointing towards i if |a_{ij}| > 1. If $a_{ij} . a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair (|a_{ij}| , |a_{ji}|) of integers.

Definition 1.8[23]: A Kac- Moody algebra g(A) is said to be of finite, affine or indefinite type if the associated GCM A is of finite, affine or indefinite type respectively.

Definition 1.9[15]: Let $A=(a_{ij})_{i,j=1}^{n}$ be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S(A) to be Quasi Hyperbolic (QH) type is S(A) has a proper sub diagram of hyperbolic type with n-1 vertices. The GCM A is of QH type if S(A) is of QH type. We then say that the Kac- Moody algebra g(A) is of QH type.

Definition 1.10[3]: A root α∈Δ is called real, if there exist a w ∈ W such w(α) is a simple root, and a root which is not real is called an imaginary root.

Definition 1.11[3]: A root ϒ ∈ Δ^{im} is said to be strictly imaginary if for every α ∈ Δ^{re}, either α + ϒ or α – ϒ is a root.

Definition 1.12[2]: We say that the generalized Cartan matrix A has the property SIM (more briefly: A ∈ SIM) if $\Delta^{sim}(A)=\Delta^{im}(A)$.

Definition 1.13[11]: Let $\alpha\in\Delta_+^{im}$ α is said to be purely imaginary if for any $\beta\in\Delta_+^{im}$, $\alpha+\beta\in\Delta_+^{im}$.

Definition 1.14[11]: A GCM A satisfies the purely imaginary property if $\Delta_+^{pim}(A)=\Delta_+^{im}(A)$. If A satisfies the purely imaginary property then the Kac-Moody algebra g(A) has the purely imaginary property.

Definition 1.15[2]: A is said to satisfy NC1, if there do not exist subsets S, T ⊂ { 1, …, n} such that A | S is affine or indefinite type, and A | S ∪ T is decomposable.

2. CLASSIFICATIONS OF QUASI-HYPERBOLIC KAC-MOODY ALGEBRA $QHA_2^{(1)}$:

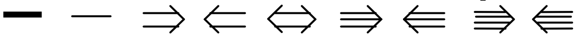
In this section, we give a complete classification of Dynkin diagrams associated with the quasi hyperbolic indefinite type of Kac-Moody algebra $QHA_2^{(1)}$ and also the properties of strictly imaginary roots and purely imaginary roots of the indefinite type of quasi hyperboplic Kac-Moody algebra $QHA_2^{(1)}$.

For the Quasi hyperbolic indefinite type of Kac-Moody algebra $QHA_2^{(1)}$, the associated GCM is

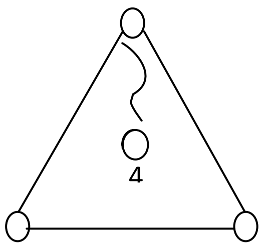
$$\begin{pmatrix} 2 & -1 & -1 & -a \\ -1 & 2 & -1 & -b \\ -1 & -1 & 2 & -c \\ -p & -q & -r & 2 \end{pmatrix},$$ where a,b,c,p,q,r are non negative integers.

Theorem 2.1(Classification Theorem):

The number of connected, non isomorphic Dynkin diagrams associated with the quasi hyperbolic indefinite type of Kac-Moody algebra $QHA_2^{(1)}$ is 212.

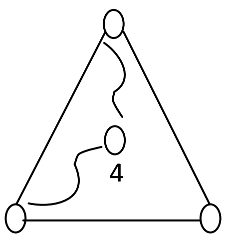
Proof:
The Dynkin diagram of the quasi hyperbolic indefinite type of Kac-Moody algebra $QHA_2^{(1)}$ is obtained by adding a fourth vertex to the Dynkin diagram of affine Kac-Moody algebra $A_2^{(1)}$ where ~ can be one of the 9 possible edges: 

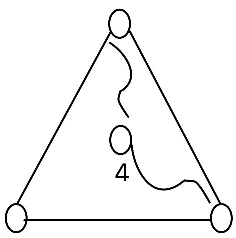
Case(i): We consider the case when a single edge is added from the fourth vertex to the Dynkin diagram of affine Kac-Moody algebra $A_2^{(1)}$.

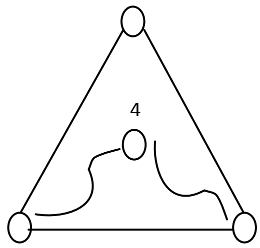


The number of Dynkin diagrams connecting the fourth vertex with any one of the other three vertices is 9 + 9 + 9 =27. Out of these 27 diagrams, excluding 5 diagrams from the hyperbolic type of rank 4 ($H_3^{(4)}$, $H_4^{(4)}$, $H_5^{(4)}$, $H_6^{(4)}$, $H_7^{(4)}$), 18 isomorphic diagrams of indefinite type of Quasi hyperbolic Kac-Moody algebras of $QHA_2^{(1)}$ and 1 diagram which is not Quasi hyperbolic type, we get 3 connected, non isomorphic Dynkin diagrams in $QHA_2^{(1)}$.

Case(ii): We consider the case now, when two edges are added from the fourth vertex to the Dynkin diagram of affine Kac-Moody algebra $A_2^{(1)}$.

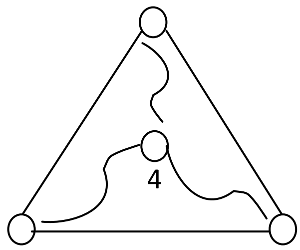






The number of Dynkin diagrams connecting the fourth vertex with any two of the three vertices by 9 possible edges as mentioned above is 243 (= 3 x 81). Out of these diagrams, excluding 1 diagram from the hyperbolic type of rank 4 ($H_2^{(4)}$), 198 isomorphic diagrams of indefinite type of Quasi hyperbolic Kac-Moody algebras of $QHA_2^{(1)}$ and 6 diagrams which are not Quasi hyperbolic type, we get 38 connected, non isomorphic Dynkin diagrams in $QHA_2^{(1)}$.

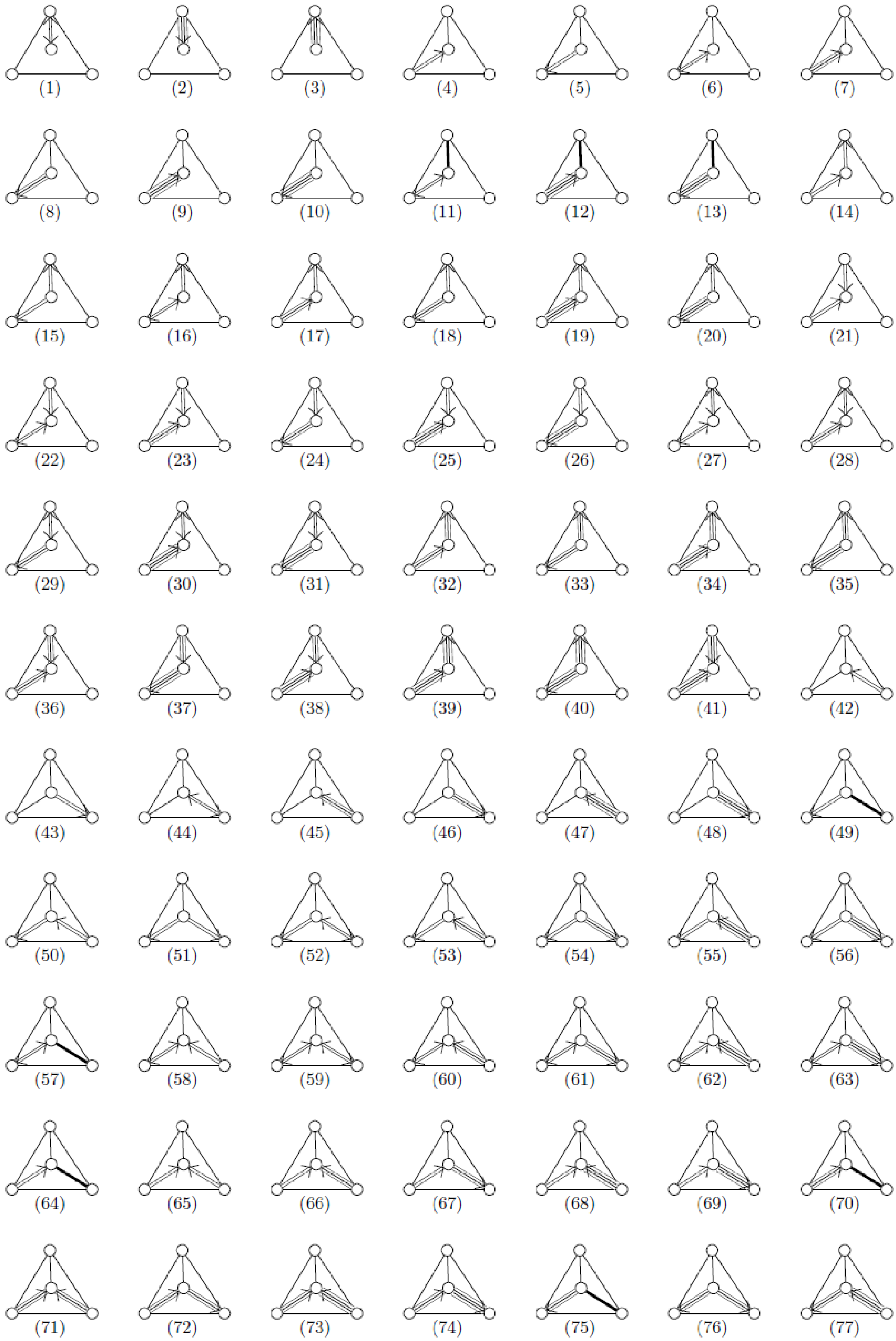
Case(iii): We consider the case now, when three edges are added from the fourth vertex to the Dynkin diagram of affine Kac-Moody algebra $A_2^{(1)}$.

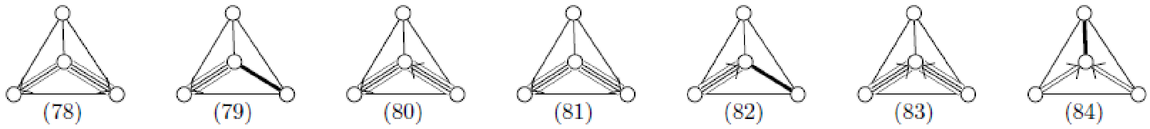


The number of Dynkin diagrams connecting the fourth vertex with all the other three vertices by 9 possible edges as mentioned above is 9^3 = 729. Out of these diagrams, excluding 1 diagram from hyperbolic type of rank 4 ($H_1^{(4)}$), 564 isomorphic diagrams of indefinite type of Quasi hyperbolic Kac-Moody algebras of $QHA_2^{(1)}$ and 10 diagrams which are not Quasi hyperbolic type, we get 154 connected, non isomorphic Dynkin diagrams in $QHA_2^{(1)}$.

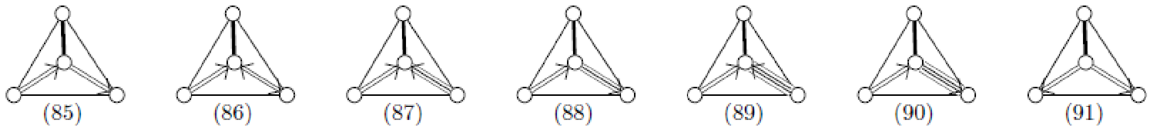
Therefore, from the above three cases we get, 3 + 38 + 154 = 195 connected, non isomorphic Dynkin diagrams of indefinite type of Quasi hyperbolic Kac-Moody algebra of $QHA_2^{(1)}$.

The Dynkin diagrams of Quasi hyperbolic Kac-Moody algebra of $QHA_2^{(1)}$ are given below:

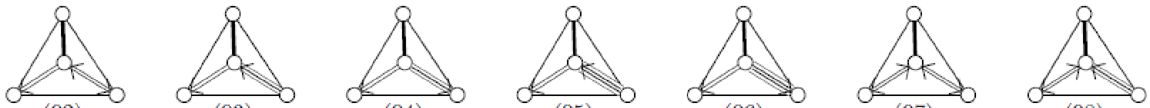

(1) (2) (3) (4) (5) (6) (7)
(8) (9) (10) (11) (12) (13) (14)
(15) (16) (17) (18) (19) (20) (21)
(22) (23) (24) (25) (26) (27) (28)
(29) (30) (31) (32) (33) (34) (35)
(36) (37) (38) (39) (40) (41) (42)
(43) (44) (45) (46) (47) (48) (49)
(50) (51) (52) (53) (54) (55) (56)
(57) (58) (59) (60) (61) (62) (63)
(64) (65) (66) (67) (68) (69) (70)
(71) (72) (73) (74) (75) (76) (77)



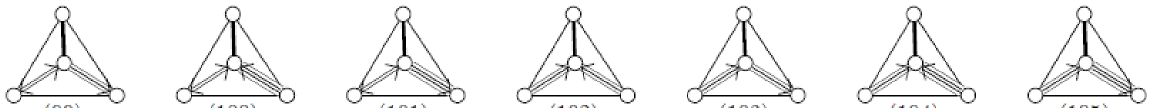
(78) (79) (80) (81) (82) (83) (84)



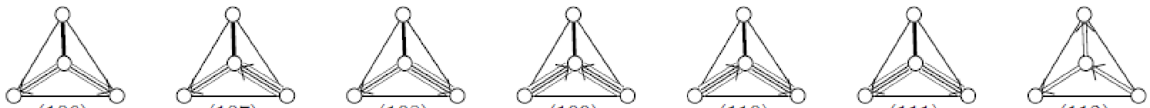
(85) (86) (87) (88) (89) (90) (91)



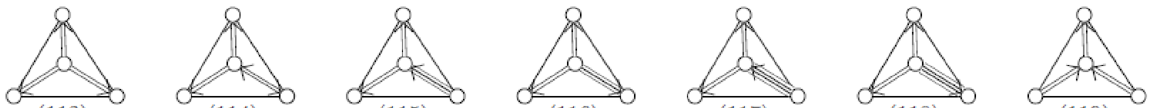
(92) (93) (94) (95) (96) (97) (98)



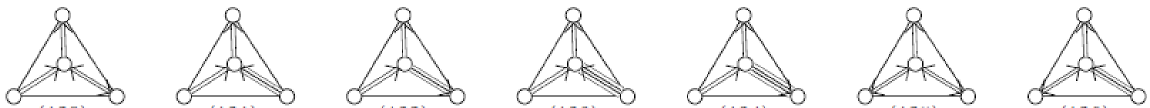
(99) (100) (101) (102) (103) (104) (105)



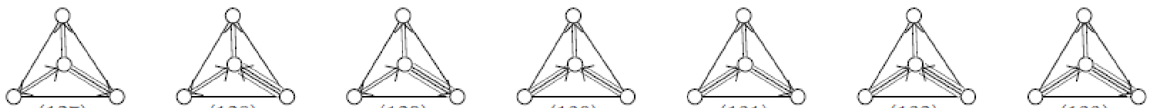
(106) (107) (108) (109) (110) (111) (112)



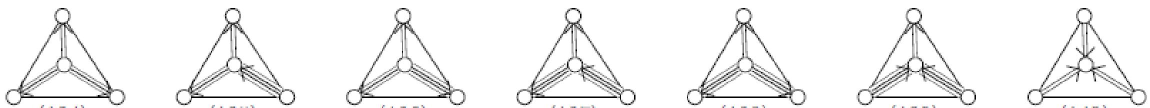
(113) (114) (115) (116) (117) (118) (119)



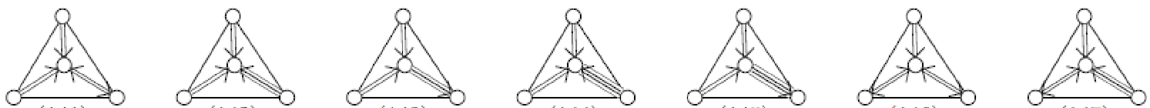
(120) (121) (122) (123) (124) (125) (126)



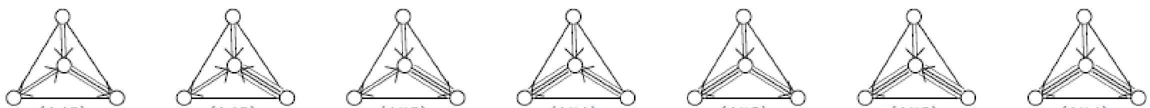
(127) (128) (129) (130) (131) (132) (133)



(134) (135) (136) (137) (138) (139) (140)



(141) (142) (143) (144) (145) (146) (147)



(148) (149) (150) (151) (152) (153) (154)

Properties of imaginary roots:

Proposition 2.2: Let $A=(a_{ij})_{i,j=1}^{4}$ be an indecomposable symmetrizable GCM associated to the Kac-Moody algebra g(A) of indefinite type of Quasi hyperbolic Kac-Moody algebra of $QHA_2^{(1)}$, then g(A) has the following properties:

(i) The imaginary roots of g(A) satisfy the purely imaginary property.

(ii) The imaginary roots of g(A) satisfy the strictly imaginary property.

Proof:

(i) Since $A=(a_{ij})_{i,j=1}^{4}$ is an indecomposable symmetrizable GCM, by using corollary 3.11 in [11], we get, $\Delta_{+}^{pim}(A)=\Delta_{+}^{im}(A)$.

Hence, g(A) has purely imaginary property.

(ii) Since $A=(a_{ij})_{i,j=1}^{4}$ is an indecomposable symmetrizable GCM, satisfies the condition given in the theorem 3.1.3 in [2], therefore A satisfies strictly imaginary property. Hence, g(A) has strictly imaginary property.

Example 1: Consider the quasi-hyperbolic Kac-Moody algebra $QHA_2^{(1)}$ with the symmetrizable

$$A=\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -4 & 0 & 0 & 2 \end{pmatrix}$$

is symmetrizable. Hence A=DB where

$$D=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad and \quad B=\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1/2 \end{pmatrix}$$

Here $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = (\alpha_3,\alpha_3) = 2$, $(\alpha_4,\alpha_4) = 1/2$, $(\alpha_1,\alpha_2) = (\alpha_1,\alpha_3) = (\alpha_1,\alpha_4) = (\alpha_2,\alpha_1) = (\alpha_2,\alpha_3) = (\alpha_3,\alpha_1) = (\alpha_3,\alpha_2) = (\alpha_4,\alpha_1) = -1$, $(\alpha_2,\alpha_4) = (\alpha_4,\alpha_2) = (\alpha_3,\alpha_4) = (\alpha_4,\alpha_3) = 0$.

Let $\alpha=2\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4$, then $(\alpha, \alpha)=-6<0$. Therefore, α is an imaginary root. Hence α is also a purely imaginary root. Let $\beta=\alpha_1+\alpha_2+\alpha_3$, then $(\beta, \beta)=0$. Therefore, β is an imaginary root. For every real root γ we can see that β+γ is a root. Hence β is a strictly imaginary root.

Example 2: Consider the quasi-hyperbolic Kac-Moody algebra $QHA_2^{(1)}$ associated with the symmetric GCM A= $\begin{pmatrix} 2 & -1 & -1 & -2 \\ -1 & 2 & -1 & -2 \\ -1 & -1 & 2 & -2 \\ -2 & -2 & -2 & 2 \end{pmatrix}$.

Here $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = (\alpha_3,\alpha_3) = (\alpha_4,\alpha_4) = 2$, $(\alpha_1,\alpha_2) = (\alpha_1,\alpha_3) = (\alpha_1,\alpha_4) = (\alpha_2,\alpha_1) = (\alpha_2,\alpha_3) = (\alpha_3,\alpha_1) = (\alpha_3,\alpha_2) = -1$, $(\alpha_4,\alpha_1) = (\alpha_2,\alpha_4) = (\alpha_4,\alpha_2) = (\alpha_3,\alpha_4) = (\alpha_4,\alpha_3) = -2$.

Let $\alpha=2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4$, then $(\alpha, \alpha)=-22<0$. Therefore, α is an imaginary root. Hence α is also a purely imaginary root. Let $\beta=\alpha_1+\alpha_2+\alpha_3$, then $(\beta, \beta)=0$. Therefore, β is an imaginary root. For every real root γ we can see that β+γ is a root. Hence β is a strictly imaginary root.

Example 3: Consider the quasi-hyperbolic Kac-Moody algebra $QHA_2^{(1)}$ associated with the symmetric GCM A= $\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -a \\ -1 & -1 & -a & 2 \end{pmatrix}$, $a > 2$.

Here $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = (\alpha_3,\alpha_3) = (\alpha_4,\alpha_4) = 2$, $(\alpha_1,\alpha_2) = (\alpha_1,\alpha_3) = (\alpha_1,\alpha_4) = (\alpha_2,\alpha_1) = (\alpha_2,\alpha_3) =(\alpha_2,\alpha_4) = (\alpha_3,\alpha_1) = (\alpha_3,\alpha_2) = (\alpha_4,\alpha_1) = (\alpha_4,\alpha_2) = -1$, $(\alpha_3,\alpha_4) = (\alpha_4,\alpha_3) = -a$.

Let $\alpha= \alpha_1+\alpha_2+2\alpha_3+2\alpha_4$, then $(\alpha, \alpha)= 2 - 8a < 0$ (since a>0). Therefore, α is an imaginary root. Hence α is also a purely imaginary root. Let $\beta=\alpha_3+\alpha_4$, then $(\beta, \beta)=0$. Therefore, β is an imaginary root. For every real root γ we can see that β+γ is a root. Hence β is a strictly imaginary root.

3. CONCLUSIONS

In this paper, a complete classification of the Dynkin diagrams of a particular type of quasi-hyperbolic Kac-Moody algebra $QHA_2^{(1)}$ is given. For the same, the properties of strictly imaginary and purely imaginary roots were also studied. We can extend this study to determine the structure and to compute the root multiplicities for $QHA_2^{(1)}$.

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