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Stress-Strength Reliability of type II compound Laplace distribution

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Abstract - *In the context of reliability, the stress-strength model describes the life of a component which has a random strength Y and is subjected to a random stress X. The component fails if the stress applied to it exceeds the strength, and the component will function satisfactorily whenever X <Y. Currently there is lot of interest in the area of stress-strength models, particularly in the estimation of the reliability R = Pr(X < Y), when X and Y are independent random variables belonging to the same univariate family of distributions. In this paper we study the stress-strength reliability R for type II compound Laplace distribution, which is obtained by compounding a Laplace distribution with the gamma distribution. Maximum likelihood estimation procedure is used to estimate the three parameters and reliability R. Finally, simulation studies were performed to validate the algorithm and also we discuss some of the applications of the model.*

***Key Words*: Laplace distribution, Maximum likelihood estimator, Reliability, Stress- Strength model, Type II Compound Laplace distribution.**

1.INTRODUCTION

Stress-strength analysis is an area in reliability theory where we assess the impact of stress on strength of devices and systems. It is measured by the expression R = Pr(X < Y) giving the reliability of a component in terms of the probability that the random variable X representing stress experienced by the component less than Y that represents the strength of the component. The component fails when the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X < Y$. Thus, $R = Pr(X < Y)$ is a measure of component reliability. The parameter R is referred to as the reliability parameter. This type of functional can be of practical importance in many applications. This measure of reliability is widely used in civil, mechanical, and aerospace engineering.

It may be noted that R has more interest than just a reliability measure. It can be used as a general measure of difference between two populations such as treatment group and control group in bio-statistical contexts and clinical trials. For instance, if X is the response for a control group, and Y refers to a treatment group, $Pr(X < Y)$ is a measure of the effect of the treatment. $R = Pr(X < Y)$ can also be useful when estimating heritability of a genetic trait. For more applications of R, see [5], [14] and [1]. In fact, Bamber gives a geometrical interpretation of $A(X; Y) = Pr(X < Y) + 2Pr(X = Y)$ and demonstrates that $A(X; Y)$ is a useful measure of the size of the difference between two populations.

1.1 Review of Literature

In most of the work in the evaluation of R = Pr(X < Y) it is assumed that both random variables has the same family and are independent. This problem has been extensively studied by various authors. Weerahandi and Johnson([16]) proposed inferential procedures for $Pr(X > Y)$ assuming that strength X and stress Y are independent normal random variables. The application of skew normal distribution to stress-strength model is illustrated in [4]. Reliability studies of the Laplace distributions were discussed in [10]. Detailed description of stress-strength theory is given in [7]. Refer [8] for generalized exponential distribution, [9] for Weibull distribution, [12] for a scaled Burr Type X distribution, [13] for 3-parameter generalized exponential distribution, [3] for Marshall-Olkin extended Lomax distribution. In [2] stress-strength reliability of the double Lomax distribution is studied and present its application to the IQ score data set from Roberts ([15]).

Laplace distributions arise as tractable lifetime models in many areas, including life testing and telecommunications. Type I and type II compound Laplace distributions were introduced in [6]. The present study deals with the reliability R of the type II compound Laplace density.

2. TYPE II COMPOUND LAPLACE DISTRIBUTION

The Laplace random variable can be regarded as the difference of two i.i.d. exponential random variables. The (symmetric) type II compound Laplace distribution (CL) introduced in [6], results from compounding a Laplace distribution with a gamma distribution. Now we derive the probability density function of type II compound Laplace from the classical Laplace (double exponential) distribution. Let X follows a classical Laplace distribution given **s** with density given by

$$f(x|s) = \frac{s}{2} e^{-s|x-\theta|}, \quad X \in R, \tag{1}$$

and let s follow a Gamma(α, β) distribution with density

$$f(s;\alpha,\beta) = \frac{s^{\alpha-1}e^{-s/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, \alpha > 0, \beta > 0, s > 0. \quad (2)$$

Then the unconditional distribution of X is the type II compound Laplace distribution with parameters (θ, α, β), denoted by X ~CL(θ, α, β) and the density function is given by

$$f(x) = \frac{\alpha\beta}{2}[1 + |x-\theta|\beta]^{-(\alpha+1)}, \alpha > 0, \beta > 0, \theta \in R, \quad X \in R. \quad (3)$$

The parameters (θ, α, β) are the location, shape and scale parameters, respectively. The cumulative density function (cdf), is given by

$$F(x) = \begin{cases} 1 - \frac{1}{2}[1 + (x-\theta)\beta]^{-\alpha}, for\ X > \theta, \\ \frac{1}{2}[1 - (x-\theta)\beta]^{-\alpha}, \quad for\ X \le \theta. \end{cases} \quad (4)$$

The density plot of Type II compound Laplace distribution for various values of θ, α, β is shown in fig. 2.1.

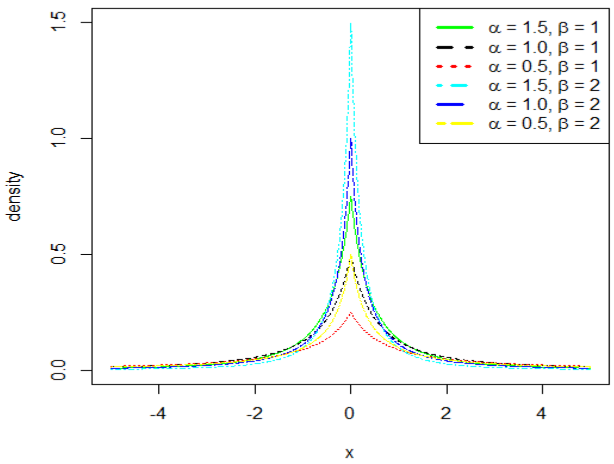

Figure 2.1. Type II compound Laplace density functions for θ = 0 and various values of α and β.

The survival function (sf) of the type II compound Laplace distribution is given by

$$S(t) = \begin{cases} \frac{1}{2}[1 + (t-\theta)\beta]^{-\alpha}, for\ T > \theta, \\ 1 - \frac{1}{2}[1 - (t-\theta)\beta]^{-\alpha}, \quad for\ T \le \theta. \end{cases} \quad (5)$$

The q^{th} quantile function (qf) of CL distribution is,

$$\xi_q = \begin{cases} \theta + \frac{1}{\beta}\left[1 - \frac{1}{(2q)^{1/\alpha}}\right], for\ q \in \left(0, \frac{1}{2}\right], \\ \theta + \frac{1}{\beta}\left[\frac{1}{(2(1-q))^{1/\alpha}} - 1\right], for\ q \in \left(\frac{1}{2}, 1\right). \end{cases} \quad (6)$$

The cdf and qf can be useful for goodness-of-fit and simulation purposes. For q $=\frac{1}{2}$, the q^{th} quantile is given by $\xi_q = \theta$. Hence, the estimate of the location parameter is given by $\hat{\theta} = \xi_{\frac{1}{2}}$, median.

The moments of type II compound Laplace distribution is given by

$$E(X-\theta)^r = \begin{cases} \frac{\alpha}{\beta^r}B(r+1, \alpha - r), for\ r\ even\ and\ 0 < r < \alpha, \\ 0, \quad r\ odd. \end{cases} \quad (7)$$

For r=1, $E(X-\theta) = 0$, hence mean(X) $= \theta$. Variance is obtained by putting r=2 and is given by

$$Var(X) = \frac{2}{\beta^2(\alpha-1)(\alpha-2)}, for\ \alpha > 2.$$

Now we derive the stress-strength reliability R for the type II compound Laplace distribution.

3. STRESS-STRENGTH RELIABILITY OF TYPE II COMPOUND LAPLACE DISTRIBUTION

Let X and Y are two continuous and independent random variables. Let f_1, F_1, f_2 and F_2 denote the probability density function (pdf) and cumulative distribution functions (cdf) of X and Y respectively. Then the reliability R can be given as,

$$R = Pr(X < Y)$$

$$= \int_{-\infty}^{\infty} Pr(X < z)Pr(Y = z)dz$$

$$= \int_{-\infty}^{\infty} F_1(z)f_2(z)\,dz \quad (8)$$

Now we evaluate the $Pr(X < Y)$ for two independent type II compound Laplace (CL) distributions. Let X and Y are continuous and independent variables having CL distribution with parameters θ_i, α_i and β_i, i = 1, 2 respectively. The pdf and cdf of CL is given in eq. (3) and eq. (4). From eq. (8) we get the reliability R for the type II compound Laplace distribution as follows.

For $\theta_1 < \theta_2$, R can be expressed as

$$R_{\theta_1<\theta_2}=\frac{\alpha_1\beta_1}{4}\left\{\int_{-\infty}^{\theta_1}[1-\beta_1(z-\theta_1)]^{-(\alpha_1+1)}[1-\beta_2(z-\theta_2)]^{-\alpha_2}dz+\int_{\theta_2}^{\theta_1}[1+\beta_1(z-\theta_1)]^{-(\alpha_1+1)}[1-\beta_2(z-\theta_2)]^{-\alpha_2}dz\right\}$$
$$+\frac{\alpha_1\beta_1}{2}\int_{\theta_2}^{\infty}[1+\beta_1(z-\theta_1)]^{-(\alpha_1+1)}\left[1-\frac{1}{2}(1+\beta_2(z-\theta_2))\right]^{-\alpha_2}dz$$

For $\theta_1 > \theta_2$, R can be expressed as

$$R_{\theta_1>\theta_2}=\frac{\alpha_1\beta_1}{4}\int_{-\infty}^{\theta_2}[1-\beta_1(z-\theta_1)]^{-(\alpha_1+1)}[1-\beta_2(z-\theta_2)]^{-\alpha_2}dz$$
$$+\frac{\alpha_1\beta_1}{2}\left\{\int_{\theta_2}^{\theta_1}[1-\beta_1(z-\theta_1)]^{-(\alpha_1+1)}\left[1-\frac{1}{2}(1+\beta_2(z-\theta_2))\right]^{-\alpha_2}dz\right\}$$
$$+\int_{\theta_1}^{\infty}[1+\beta_1(z-\theta_1)]^{-(\alpha_1+1)}\left[1-\frac{1}{2}(1+\beta_2(z-\theta_2))\right]^{-\alpha_2}dz$$

Thus, the reliability parameter R can be expressed as

$$R = R_{\theta_1<\theta_2} I_{\theta_1<\theta_2} + R_{\theta_1>\theta_2} I_{\theta_1>\theta_2}, \qquad (9)$$

where $I(.)$ is the indicator function. The MLE of the $R = P(X > Y)$ can be obtained by replacing the parameters $\theta_1, \alpha_1, \beta_1, \theta_2, \alpha_2$ and β_2 in the expression of R in the eq. (9) by their MLE's. Using Maple program we can evaluate the integrals and compute the maximum likelihood estimator of R.

3. ESTIMATION OF TYPE II COMPOUND LAPLACE DISTRIBUTION

In this section we study the problem of estimating three unknown parameters, $\Theta = (\theta, \alpha, \beta)$, of CL distribution. The estimate of θ is median. The method of moments or maximum likelihood estimation method can be employed to estimate Θ as described below. Let X $=(X_1, X_2, \dots, X_n)$ be independent and identically distributed samples from type II compound Laplace distribution with parameters Θ. To estimate Θ under the method of moments, four first moments, E (X^r); r =1, 2, 3 are equated to the corresponding sample moments and the resulted system of equations are solved for the unknown parameters. These moments can be obtained from Eq. (7) but they exist only when $\alpha > 3$. An alternative method is a maximum likelihood estimation where the likelihood function is maximized to estimate the unknown parameters and is describe below.

The log-likelihood function of the data X takes the form,

$$\log L(\Theta; X) = n\log\alpha + n\log\beta - (\alpha+1)S(\theta,\beta).$$

Here,
$S(\theta,\beta) = \sum_{i=1}^{n} S_i(\theta,\beta) = \sum_{i=1}^{n} \log(1 + \beta(x_i-\theta)^+ + \beta(x_i-\theta)^-)$

Where, $(x_i-\theta)^+ = x_i - \theta$; if $x_i > \theta$, and = 0 otherwise, and $(x_i-\theta)^-$ = - $(x_i - \theta)$; if $x_i \le \theta$, and = 0 otherwise.

The MLEs of (α, β) for given $\theta = \hat{\theta}$ are obtained by solving the score equations for α and β. This leads to the following equations which are solved iteratively.

$$\hat{\alpha} = \frac{n}{S(\hat{\theta},\hat{\beta})}$$

$$\frac{n}{\hat{\beta}} = (\hat{\alpha}+1)\sum_{i=1}^{n}\frac{(x_i-\theta)^+ + (x_i-\theta)^-}{(1+\hat{\beta}(x_i-\theta)^+ + \hat{\beta}(x_i-\theta)^-)}$$

The maximization of the likelihood can be implemented using the **optim** function of the R statistical software, applying the BFGS algorithm (see [11]). Estimates of the standard errors were obtained by inverting the numerically differentiated information matrix at the maximum likelihood estimates.

4. SIMULATION

In this section we use the simulation study of the CL distribution to validate the estimation algorithm developed in R. Since we can express the distribution function of the CL distribution as well as its inverse in closed form, the inversion method of simulation is straightforward to implement. We simulated a data set of size n from the CL distribution with parameters $(\theta_1, \alpha_1, \beta_1)$ = (0.05, 1, 0.05) and $(\theta_2, \alpha_2, \beta_2)$ = (1, 4, 0.05) by inverting the distribution function given by eq. (4) in R package and then using the algorithm developed in R to obtain the maximum likelihood estimates and standard error for the parameters.

Table -1: Simulation study- parameter values used for simulation (true), MLE and standard errors (SE) for the parameters

n	θ_1	α_1	β_1	θ_2	α_2	β_2	R
100	0.053 (0.011)	1.23 (0.531)	0.054 (0.031)	0.992 (0.009)	4.236 (0.652)	0.053 (0.004)	0.612 (0.571)
200	0.048 (0.005)	1.171 (0.313)	0.052 (0.003)	1.005 (0.004)	4.209 (0.125)	0.051 (0.002)	0.647 (0.348)

500	0.051 (0.003)	1.103 (0.373)	0.052 (0.002)	0.996 (0.001)	4.121 (0.101)	0.051 (0.002)	0.689 (0.327)
1000	0.049 (0.001)	1.093 (0.213)	0.051 (0.001)	0.999 (0.002)	4.103 (0.065)	0.049 (0.001)	0.714 (0.447)

5. APPLICATIONS

In reliability studies, we are interested in calculating the probability of the event $X < Y$, where X and Y are independent random variables. One of these situations is in a stress-strength model where X and Y are assumed respectively as strength and stress random variables. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the aging of concrete pressure vessels. In engineering science, safety margin (SM) and safety factor (SF) are functions of stress and strength and they are respectively settled as $SM = Y - X$ and $SF = Y/X$ and the most interest concerns in obtaining the probability $p = Pr(SM > 0) = Pr(Y - X > 0)$ or $Pr(SF > 1) = Pr(Y/X > 1)$. The practical applications no means confined to engineering or to military problems but also to medical statistics and other areas. Pr(X < Y) is of greater interest than just in reliability since it provides a general measure of the difference between two populations and has applications in many areas. One of the interesting applications is the relationship between the stress-strength models and the quality control concept, such as process capability indices.

6. CONCLUSIONS

R = Pr(X < Y) is of greater interest than just in reliability since it provides a general measure of the difference between two populations and has applications in many areas. The derived the stress-strength reliability R of type II compound Laplace distributions has many applications in the field engineering, medicine and genetics.

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