Geometric construction of the action for pure supergravity coupled to Wess-Zumino multiplets

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Abstract - *In this article we build "step-by-step" the complete Lagrangian relative to the coupling of the scalar multiplets, or Wess-Zumino multiplets, with the action of pure D = 4, N = 1 supergravity. We follow the so-called "geometric approach", i.e. using the concepts of supersymmetry, superspace, rheonomic principle and considering all fields as superforms in superspace.*

Key Words: *Action principle, Lagrangian, Supergravity, Wess-Zumino multiplets, Supersymmetry, Superspace, Rheonomic principle, Differential Geometry.*

1. INTRODUCTION

In physics, in particular in the Hamiltonian and Lagrangian mechanics, the "action" is a scalar which has the dimensions of an energy times a time. It is a tool that allows to study the motion of a dynamical system, and it is used in classical mechanics, electromagnetism, relativistic mechanics and quantum mechanics.
Usually the action corresponds to an integral over time and possibly with respect to a set of spatial variables; sometimes the integral is performed along the curve traveled by the considered system in the configuration space. In Lagrangian and Hamiltonian mechanics it is usually defined as the integral over time of a characteristic function of the considered mechanical system, the "Lagrangian", evaluated between the initial and final times of the temporal evolution of the system between two positions.
The main motivation in defining the concept of action lies in the variational principle of Hamilton, according to which every mechanical system is characterized by the fact that its temporal evolution between two positions in space minimizes the action. Such statement is expressed by saying that the temporal evolution of a physical system between two instants of the configuration space is a stationary point for the action, usually a minimum point, for small perturbations of the traveled path. The variational principle allows in this way to reformulate the equations of motion, typically differential equations, through an equivalent integral equation. It is then possible to write the equations of motion. In describing physical systems, the invariance (symmetry) of the Lagrangian with respect to continuous transformations of coordinates determines the presence of conserved quantities during the motion, i.e. of constants of motion, in accordance with the Noether theorem.
In the following paragraphs we will build "step-by-step" the action related to the scalar multiplets, or Wess-Zumino multiplets, to be coupled with that of pure supergravity. The work is performed in the so called "geometric approach", i.e. considering all fields as superforms in superspace [1]. In Paragraphs 2, 3, 4 we explicitly build the various parts of the action, arriving in Paragraph 5 to write the expression of the complete Lagrangian of pure supergravity coupled with Wess-Zumino multiplets.

2. THE CONSTRUCTION OF THE ACTION

As in pure supergravity, the principle of action for the interacting system is given by [2,3]:

$$A=\int_{M_4\subset R^{4/4}} d\ \mathcal{L}, \tag{1}$$

with $\mathcal{L}$ a 4-form built with the basic fields of the theory. Making A stationary with respect to variations in both fields and surface M_4, we obtain equations of differential forms that need to be consistent with the rheonomic parameterization already determined through the Bianchi identity [4,5].

The rules for writing the ansatz for $\mathcal{L}$ are:

1) $\mathcal{L}$ must be built using only the wedge product "$\wedge$" and the exterior derivative "d". The Hodge duality is excluded and replaced by the Hodge duality on indexes of tangent space ε_{abcd}.

2) $\mathcal{L}$ must be Lorentz invariant.

3) $\mathcal{L}$ must be invariant under general transformations of holomorphic coordinates in the Kähler manifold.

4) $\mathcal{L}$ must respect the scaling invariance of V^a, ω^{ab}, ψ, z^i, χ^i; all terms must have the same scaling weight w = 2 of the Einstein term.

5) $\mathcal{L}$ must contain the kinetic terms of all physical fields.

The most general Lagrangian that meets these requirements can be written as a sum of various sub-Lagrangians. For first we divide $\mathcal{L}$ in a part $\mathcal{L}_1$ which survive to the limit "$e \to 0$" , and a part $\Delta\mathcal{L}$ which is proporzional to "e":

$$\mathcal{L} = \mathcal{L}_1 + \Delta\mathcal{L} \,. \tag{2}$$

Also we divide $\mathcal{L}_1$ as follows:

$$\mathcal{L}_1 = \mathcal{L}_{(\text{KIN})} + \mathcal{L}_{(\text{PAULI})} + \mathcal{L}_{(\text{TORSION})} + \mathcal{L}^{(\text{4-FERMI})}{}_{(2\psi 2V)} + \mathcal{L}^{(\text{4-FERMI})}{}_{(4V)} \,, \tag{3}$$

About these parts:

a) $\mathcal{L}_{(\text{KIN})}$ contains the kinetic terms of all fields, in particular that of the scalar field, written in the first order formalism;

b) $\mathcal{L}_{(\text{PAULI})}$ contains the terms which couple the bosonic derivative dz^i to fermionic currents $\bar{\chi}^i \gamma \psi$;

c) $\mathcal{L}_{(\text{TORSION})}$ contains the terms $R^a \wedge \ldots$ such that the variation $\delta\omega^{ab}$ gives the field equation $R^c = 0$;

d) $\mathcal{L}^{(\text{4-FERMI})}{}_{(2\psi 2V)}$ contains the "not-derivative 4-Fermi" terms of the form $\chi\,\chi\,\psi \wedge \psi \wedge V \wedge V$;

e) $\mathcal{L}^{(\text{4-FERMI})}{}_{(4V)}$ contains the "not-derivative 4-Fermi" terms of the form $\chi\,\chi\,\chi\,\chi\, V \wedge V \wedge V \wedge V$. This part of the Lagrangian must be fixed by the supersymmetry invariance, that is $\underline{\varepsilon}\,\rfloor\, d\,\mathcal{L} = 0$.

We also divide $\Delta\mathcal{L}$ as follows:

$$\Delta\mathcal{L} = \Delta\mathcal{L}_{(\psi\psi VV)} + \Delta\mathcal{L}_{(\chi\psi VVV)} + \Delta\mathcal{L}_{(\chi\chi VVVV)} + \Delta\mathcal{L}_{(\text{Potential})} \,, \tag{4}$$

with:

i) $\Delta\mathcal{L}_{(\psi\psi VV)}$ is related to the mass term of gravitino, which has the coefficient linked to the auxiliary field S;

ii) $\Delta\mathcal{L}_{(\chi\psi VVV)}$ is related to the "not-diagonal" mass of spin 1/2, spin 3/2, which has the coefficient linked to the auxiliary field $\mathcal{H}^i$;

iii) $\Delta\mathcal{L}_{(\chi\chi VVVV)}$ is related to the mass term of spin 1/2, which has the coefficient linked to the derivative of the next term;

iv) $\Delta\mathcal{L}_{(\text{Potential})}$ is related to the potential term of the scalar field, which will be expressed as quadratic form in S and $\mathcal{H}^i$ [6].

3. CONSTRUCTION OF L_1

For the construction of $\mathcal{L}_1$ we consider all terms, with the exception of $\mathcal{L}^{(\text{4-FERMI})}{}_{(4V)}$, which will be fixed at the end by a supersymmetry transformation. We start with the following general ansatz:

$$\begin{aligned}
\mathcal{L}_{(\text{KIN})} =\ & \varepsilon_{abcd}\, R^{ab} \wedge V^c \wedge V^d - 4(\bar{\psi}^\bullet \wedge \gamma_a\, \rho_\bullet + \bar{\rho}^\bullet \wedge \gamma_a\, \psi_\bullet) \wedge V^a + \\
& + i\,\delta_1\, g_{ij^*}\, (\bar{\chi}^i \gamma_a \nabla\chi^{j^*} + \bar{\chi}^{j^*} \gamma_a \nabla\chi^i) \wedge V_b \wedge V_c \wedge V_d\, \varepsilon^{abcd} + \\
& + \delta_2\, g_{ij^*} Z^i{}_a\, (dz^{j^*} - \bar{\chi}^{j^*} \psi^\bullet) \wedge V_b \wedge V_c \wedge V_d\, \varepsilon^{abcd} + \\
& + \delta_3\, g_{ij^*} Z^{j^*}{}_a\, (dz^i - \bar{\chi}^i \psi_\bullet) \wedge V_b \wedge V_c \wedge V_d\, \varepsilon^{abcd} + \\
& + \delta_4\, g_{ij^*} Z^i{}_a\, Z^{j^* a}\, \varepsilon_{b_1 b_2 b_3 b_4}\, V^{b_1} \wedge V^{b_2} \wedge V^{b_3} \wedge V^{b_4} \,;
\end{aligned} \tag{5}$$

$$\begin{aligned}
\mathcal{L}_{(\text{PAULI})} =\ & i\,\delta_5\, g_{ij^*}\, dz^i\, \bar{\chi}^{j^*} \gamma_{ab}\, \psi^\bullet \wedge V^a \wedge V^b + \\
& + i\,\delta_6\, g_{ij^*}\, d\bar{z}^{j^*}\, \bar{\chi}^i \gamma_{ab}\, \psi_\bullet \wedge V^a \wedge V^b \,;
\end{aligned} \tag{6}$$

$$\mathcal{L}_{(\text{TORSION})} = \delta_7\, R^a \wedge V_a \wedge g_{ij^*}\, \bar{\chi}^i \gamma_b\, \chi^{j^*}\, V^b \,; \tag{7}$$

$$\mathcal{L}^{(\text{4-FERMI})}{}_{(2\psi 2V)} = i\,\delta_8\, g_{ij^*}\, \bar{\chi}^i \gamma_a\, \chi^{j^*}\, \bar{\psi}^\bullet \wedge \gamma_b\, \psi_\bullet \wedge V^a \wedge V^b \,. \tag{8}$$

The first two terms correspond to the action of pure supergravity; the real coefficients $\delta_1, \ldots, \delta_8$ are determined by the equations of motion. The sign of δ_1 is fixed by the requirement of positive energy:

$$\delta_1 = -\alpha\,; \quad (\alpha > 0)\,. \tag{9}$$

The "hermiticity" of the Lagrangian brings to:

$$\delta_3 = \delta_2\,; \qquad \delta_6 = -\delta_5\,. \tag{10}$$

Doing the variation in $\delta z^i{}_a$ we get:

$$\delta_4 = -\frac{1}{4}\delta_2 = -\frac{1}{4}\delta_3 . \quad (11)$$

From the variation in $\delta \chi^{j^*}$ we get a field equation from whose projections $\psi \wedge V \wedge V \wedge V$ and $\psi \wedge \psi \wedge V \wedge V$ it follows:

$$\delta_2 = 2\alpha ; \quad \delta_5 = -6\alpha ; \quad \delta_8 = 6\alpha . \quad (12)$$

The variation in $\delta \omega^{ab}$ gives:

$$\delta_7 = -3\alpha . \quad (13)$$

So we have:

$$\delta_1 = -\alpha ; \quad \delta_2 = 2\alpha ; \quad \delta_3 = 2\alpha ; \quad \delta_4 = -\frac{1}{2}\alpha ; \quad \delta_5 = -6\alpha ;$$
$$\delta_6 = 6\alpha ; \quad \delta_7 = -3\alpha ; \quad \delta_8 = 6\alpha . \quad (14)$$

Considering now the equation of gravitino (variation in $\delta \overline{\psi}^{\bullet}$) and setting:

$$A_a = \mu T_a , \quad A'_a = \frac{\nu}{2} T_a , \quad (15)$$

with T_a given by $T_a = \overline{\chi}^i \gamma_a \chi^{j^*} g_{ij^*}$, and μ and ν coefficients to be determined, the projection $\psi \wedge V \wedge V$ gives:

$$\nu = \frac{3}{2}\alpha , \quad \mu = 0 , \quad (16)$$

This is consistent with what we have imposed in the Bianchi identities [3,6]. We get now from the Bianchi identity:

$$\mathscr{D}\rho_{\bullet} + \frac{1}{4} R^{ab} \wedge \gamma_{ab} \psi_{\bullet} + \frac{1}{2} g_{ij^*} dz^i \wedge d\overline{z}^{j^*} \wedge \psi_{\bullet} = 0 . \quad (17)$$

with relations $S = i e \exp(\frac{G}{2})$ and (15), in the projection $\psi \wedge \psi \wedge \psi$:

$$\nu = \mu + \frac{1}{2} p = \frac{1}{4} . \quad (18)$$

We can then rewrite all coefficients in terms of the Kähler charge of gravitino $p = 1/2$:

$$\delta_1 = -\frac{1}{3} ; \quad \delta_2 = \frac{2}{3} ; \quad \delta_3 = \frac{2}{3} ; \quad \delta_4 = -\frac{1}{6} ; \quad \delta_5 = -2 ; \quad \delta_6 = 2 ;$$
$$\delta_7 = -1 ; \quad \delta_8 = 2 ; \quad \mu = 0 ; \quad \nu = \frac{1}{4} . \quad (19)$$

For the calculation of $\mathcal{L}^{(4\text{-FERMI})}{}_{(4V)}$ the ansatz is given by:

$$\mathcal{L}^{(4\text{-FERMI})}{}_{(4V)} = \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d \; \overline{\chi}^i \gamma_m \chi^{j^*} \; \overline{\chi}^k \gamma^m \chi^{l^*} \times$$
$$\times (m \, g_{ij^*} g_{kl^*} + n R_{j^* i l^* k}) , \quad (20)$$

and values of m and n result:

$$m = -\frac{p^2}{12} = -\frac{1}{48} ; \quad n = -\frac{p}{24} = -\frac{1}{48} . \quad (21)$$

With this the Lagrangian $\mathcal{L}_1$ is completely built.

4. CONSTRUCTION OF ΔL

From the equation of gravitino we see that, if "$e \neq 0$", the first term $-8\gamma_a \rho_{\bullet} \wedge V^a$ generates a further term:

$$8 S \gamma_{ab} \psi^{\bullet} \wedge V^a \wedge V^b . \quad (22)$$

To compensate it, the following term is added to the Lagrangian:

$$\Delta\mathcal{L}_{(\psi\psi VV)} = -4 (S \overline{\psi}^{\bullet} \wedge \gamma_{ab} \psi^{\bullet} + S^* \overline{\psi}_{\bullet} \wedge \gamma_{ab} \psi_{\bullet}) \wedge V^a \wedge V^b . \quad (23)$$

Doing that, considering the equation which corresponds to the variation in $\delta \chi^{j^*}$, for $e \neq 0$ it generates an unbalanced term that is eraseable by adding to the Lagrangian the term:

$$\Delta\mathcal{L}_{(\chi\psi VVV)} = (\mathscr{F}_{i^*} \; \overline{\chi}^{i^*} \gamma_a \psi_{\bullet} + \mathscr{F}_i \; \overline{\chi}^i \gamma_a \psi^{\bullet}) V_b \wedge V_c \wedge V_d \, \varepsilon^{abcd} , \quad (24)$$

with:

$$\mathscr{F}_i = -2 i \delta_1 g_{ij^*} \mathscr{H}^{j^*} ; \quad \mathscr{F}_{i^*} = (\mathscr{F}_i)^* . \quad (25)$$

Having introduced these terms, we have to consider also a potential term:

$$\Delta\mathcal{L}_{(\text{Potential})} = -W \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d , \quad (26)$$

with:

$$W = -2 S S^* - \frac{1}{2}\delta_1 g_{ij^*} \mathscr{H}^i \mathscr{H}^{j*}. \quad (27)$$

We have then:

$$\Delta \mathcal{L}_{(\chi\chi VVVV)} = (m_{ij} \bar{\chi}^i \chi^j + m_{i^* j^*} \bar{\chi}^i \chi^{j^*}) \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d . \quad (28)$$

From the variation in $\delta\chi^j$ of (28), from that in $\delta\psi^\bullet$ of (24) and from that in δz^i of (26), we get the condition:

$$\partial_i W - 2 m_{ij} \mathscr{H}^j + \mathscr{F}_i S^* = 0. \quad (29)$$

Considering therefore that:

$$S = i e \exp\left[\frac{1}{2} G(z,\bar{z})\right], \quad (30)$$

$$\mathscr{H}^i = 2 e (g^{ij^*} \partial_{j^*} G) \exp\left[\frac{1}{2} G(z,\bar{z})\right], \quad (31)$$

$$W = -\frac{2}{3} e^2 (3 - g^{ij^*} \partial_i G\, \partial_{j^*} G) \exp\left[\frac{1}{2} G(z,\bar{z})\right], \quad (32)$$

$$\mathscr{F}_i = \frac{4}{3} i e \partial_i G \exp\left[\frac{1}{2} G(z,\bar{z})\right], \quad (33)$$

they allow to obtain from (29):

$$m_{ij} = \frac{e}{6} (\partial_i G\, \partial_j G + \nabla_i \partial_j G) \exp\left[\frac{1}{2} G(z,\bar{z})\right]. \quad (34)$$

5. EXPRESSION OF THE COMPLETE LAGRANGIAN OF PURE SUPERGRAVITY COUPLED WITH WESS-ZUMINO MULTIPLETS

Thank to all quantities obtained in the previous sections, we are now able to write a complete Lagrangian of the pure supergravity coupled with n scalar multiplets. The complete expression is as follows:

$$\mathcal{L}^{(SUGRA+WZ)} = \varepsilon_{abcd} R^{ab} \wedge V^c \wedge V^d - 4(\bar{\psi}^\bullet \wedge$$

$$\wedge \gamma_a \rho_\bullet + \bar{\rho}^\bullet \wedge \gamma_a \psi_\bullet) \wedge V^a - \frac{i}{3} g_{ij^*} (\bar{\chi}^i \gamma_a \nabla \chi^{j^*} +$$

$$+ \bar{\chi}^{j^*} \gamma_a \nabla \chi^i) \wedge V_b \wedge V_c \wedge V_d \varepsilon^{abcd} + \frac{2}{3} g_{ij^*} (Z^i_a (d\bar{z}^{j^*} -$$

$$- \bar{\chi}^{j^*} \psi^\bullet) + \bar{Z}^{j^*}_a (dz^i - \bar{\chi}^i \psi_\bullet)) \wedge V_b \wedge V_c \wedge V_d \varepsilon^{abcd} -$$

$$- \frac{1}{6} g_{ij^*} Z^i{}_a Z^{j^* a} \varepsilon_{b_1 b_2 b_3 b_4} V^{b_1} \wedge V^{b_2} \wedge V^{b_3} \wedge V^{b_4} -$$

$$- 2 i g_{ij^*} (dz^i \wedge \bar{\chi}^{j^*} \gamma_{ab} \psi^\bullet - d\bar{z}^{j^*} \wedge \bar{\chi}^i \gamma_{ab} \psi_\bullet) \wedge V^a \wedge V^b -$$

$$- R^a \wedge V_a \wedge g_{ij^*} \bar{\chi}^i \gamma_b \chi^{j^*} V^b + 2 g_{ij^*} \bar{\chi}^i \gamma_a \chi^{j^*} \bar{\psi}^\bullet \wedge$$

$$\wedge \gamma_b \psi_\bullet \wedge V^a \wedge V^b - \frac{1}{48} \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d \bar{\chi}^i \gamma_m \chi^{j^*}$$

$$\bar{\chi}^k \gamma^m \chi^{l^*} (g_{ij^*} g_{kl^*} + R_{j^* i l^* k}) - -4(S \bar{\psi}^\bullet \wedge \gamma_{ab} \psi^\bullet +$$

$$+ S^* \bar{\psi}_\bullet \wedge \gamma_{ab} \psi_\bullet) \wedge V^a \wedge V^b + (\mathscr{F}_{i^*} \bar{\chi}^{i^*} \gamma_a \psi_\bullet +$$

$$+ \mathscr{F}_i \bar{\chi}^i \gamma_a \psi^\bullet) \wedge V_b \wedge V_c \wedge V_d \varepsilon^{abcd} + (m_{ij} \bar{\chi}^i \chi^j +$$

$$+ m_{i^* j^*} \bar{\chi}^i \chi^{j^*} - W) \varepsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d , \quad (35)$$

with S, W, $\mathscr{F}_i$ and m_{ij} given by relations (30) and (32-34).

6. CONCLUSIONS

In this paper we have explicitly built “step-by-step” the complete Lagrangian related to the coupling of the scalar multiplets, or Wess-Zumino multiplets, to the action of pure $N = 1$ supergravity in 4 dimensions. It has been followed the geometric approach, considering all fields as superforms in the superspace. The treatment is exhaustive and very elegant from a mathematical point of view. Considering also the vector multiplets, we can get the complete Lagrangian of supergravity coupled to matter. Supergravity theories are effective theories of superstring theories; they are considered, with quantum gravity, one of the approaches of contemporary high energy physics for a unified theory of forces of Nature known at today [7-9].

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