

Fixed point theory for cyclic generalized contractions in partial metric spaces

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Abstract

In this article, we give some fixed point theorems for mappings satisfying cyclical generalized contractive conditions in complete partial metric spaces.

Key Words:Fixed point, partial metric, (c)-comparison function, special Picard operator.

1 INTRODUCTION:

The well known Banach's fixed point theorem asserts that: If (X, d) is a complete metric space and $f: X \rightarrow X$ is a mapping such that

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$ and some $\lambda \in [0, 1)$, then f has a unique fixed point in X . Kannan [1] extended Banach's fixed point theorem to the class of maps $f: X \rightarrow X$ satisfying the following contractive condition:

$$d(f(x), f(y)) \leq \lambda [d(x, f(x)) + d(y, f(y))]$$

for all $x, y \in X$ and some $\lambda \in (0, 1/2)$. Reich [2] generalized both results using the contractive condition:

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta d(x, f(x)) + \gamma d(y, f(y))$$

for each $x, y \in X$, where α, β, γ are nonnegative real numbers satisfying $\alpha + \beta + \gamma < 1$.

Matkowski [3] used the following contractive condition:

$$d(f(x), f(y)) \leq \varphi(d(x, y))$$

for all $x, y \in X$, where $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$ for all $t > 0$.

In 1994, Matthews [4] introduced the notion of a partial metric space and obtained a generalization of Banach's fixed point theorem for partial metric spaces. Recently, Altun et al. [5] (see also Altun and Sadarangani [6]) gave some generalized versions of the fixed point theorem of Matthews [4]. Di Bari and Vetro [7] obtained some results concerning cyclic mappings in the framework of partial metric spaces. We recall below the definition of partial metric space and some of its properties (see [4, 5, 8, 9]).

Definition 1 A partial metric on a nonempty set X is a function $p: X \times X \rightarrow \mathbb{R}_+$ such that for all $x, y, z \in X$:

$$p_1: x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$$

$$p_2: p(x, x) \leq p(x, y),$$

$$p_3: p(x, y) = p(y, x),$$

$$p_4: p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

A partial metric space is a pair (X, p) where X is a nonempty set and p is a partial metric on X . The function $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}_+$ defines a partial metric on \mathbb{R}_+ . Other interesting examples of partial metric spaces can be found in [4, 10, 11]. It is known [8] that each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p -balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X , then the function $p^s : X \times X \rightarrow \mathbb{R}_+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

defines a metric on X (see [12]).

Let (X, p) be a partial metric space.

A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ [4, 5, 8] if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$.

A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence [4, 5, 8] if there exists (and is finite) $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete [4, 5, 8] if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

It is evident that every closed subset of a complete partial metric space is complete.

Lemma 2 [4, 5, 8] *Let (X, p) be a partial metric space.*

(1) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s) .

(2) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$$

Definition 3 [13] *Let X be a nonempty set, m a positive integer and $f : X \rightarrow X$ an operator. By definition, $X = \bigcup_{i=1}^m X_i$ is a cyclic representation of X with respect to f if*

(i) $X_i, i = 1, \dots, m$ are nonempty sets;

(ii) $f(X_1) \subset X_2, \dots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1$.

Definition 4 [13] A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a comparison function if it satisfies:

- (i) φ is monotone increasing, i.e., $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$, for any $t_1, t_2 \in \mathbb{R}_+$;
- (ii) $(\varphi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \rightarrow \infty$ for all $t \in \mathbb{R}_+$.

Definition 5 [13] A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a (c)-comparison function if it satisfies:

- (i) φ is monotone increasing;
- (ii) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k, \text{ for } k \geq k_0 \text{ and any } t \in \mathbb{R}_+.$$

Lemma 6 [13] If $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a (c)-comparison function, then the following hold:

- (i) φ is a comparison function;
- (ii) $\varphi(t) < t$, for any $t \in \mathbb{R}_+$;
- (iii) φ is continuous at 0;
- (iv) the series $\sum_{k=0}^{\infty} \varphi^k(t)$ converges for any $t \in \mathbb{R}_+$.

In this article, we prove some fixed point theorems for generalized contractions defined on cyclic representation in the setting of partial metric spaces.

2 MAIN RESULTS:

Definition 7 Let (X,p) be a partial metric space. A mapping $f : X \rightarrow X$ is called a φ -contraction if there exists a comparison function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$p(f(x),f(y)) \leq \varphi(p(x,y)) \text{ for all } x, y \in X.$$

Definition 8 Let (X, p) be a partial metric space, m a positive integer, A_1, \dots, A_m nonempty closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. An operator $f : Y \rightarrow Y$ is called a cyclic φ -contraction if

- (i) $\bigcup_{i=1}^m A_i$ is a cyclic representation of Y w.r.t f ;
- (ii) There exists a (c)-comparison function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$(2.1) \quad p(f(x),f(y)) \leq \varphi(p(x,y)) \text{ for any } x \in A_i, y \in A_{i+1}, \text{ where } A_{m+1} = A_1.$$

Theorem 9 Let (X, p) be a complete partial metric space, m a positive integer, A_1, \dots, A_m closed nonempty subsets of $X, Y = \bigcup_{i=1}^m A_i, \varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a (c) -comparison function and $f: Y \rightarrow Y$ an operator. Assume that

- (i) $\bigcup_{i=1}^m A_i$ is a cyclic representation of Y w.r.t f ;
- (ii) f is a cyclic φ -contraction.

Then f has a unique fixed point $x^* \in \bigcup_{i=1}^m A_i$ and the Picard iteration $\{x_n\}$ converges to x^* for any initial point $x_0 \in Y$.

Proof. Let $x_0 \in Y = \bigcup_{i=1}^m A_i$, and set $x_n = f(x_{n-1}), n \geq 1$.

For any $n \geq 0$ there is $i_n \in \{1, \dots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$. Then by (2.1) we have $p(x_n, x_{n+1}) = p(f(x_{n-1}), f(x_n)) \leq \varphi(p(x_{n-1}, x_n))$.

Since φ is monotone increasing, we get by induction that

$$(2.2) \quad p(x_n, x_{n+1}) \leq \varphi^n(p(x_0, x_1)).$$

By definition of φ , thus letting $n \rightarrow \infty$ in (2.2), we obtain that

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0.$$

On the other hand, since

$$p(x_n, x_n) \leq p(x_n, x_{n+1}) \text{ and } p(x_{n+1}, x_{n+1}) \leq p(x_n, x_{n+1}),$$

then from (2.2) we have

$$(2.3) \quad p(x_n, x_n) \leq \varphi^n(p(x_0, x_1)) \text{ and } p(x_{n+1}, x_{n+1}) \leq \varphi^n(p(x_0, x_1)).$$

Thus, we have

$$p^s(x_n, x_{n+1}) \leq 4\varphi^n(p(x_0, x_1)).$$

Since φ is a (c) -comparison function, from Lemma 6, it follows that

$$\lim_{n \rightarrow \infty} p^s(x_n, x_{n+1}) = 0.$$

So for $k \geq 1$, we have

$$p^s(x_n, x_{n+k}) \leq p^s(x_n, x_{n+1}) + \dots + p^s(x_{n+k-1}, x_{n+k}) \leq 4 \sum_{m=n}^{n+k-1} \varphi^m(p(x_0, x_1)).$$

Again since φ is a (c) -comparison function, by Lemma 6, it follows that

$$\sum_{m=0}^{\infty} \varphi^m(p(x_0, x_1)) < \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in the metric subspace (Y, p^s) . Since Y is closed, the subspace (Y, p) is complete. Then from Lemma 2, we have that (Y, p^s) is complete. Let $\lim_{n \rightarrow \infty} p^s(x_n, y) = 0$.

Now Lemma 2 further implies that

$$(2.4) \quad p(y, y) = \lim_{n \rightarrow \infty} p(x_n, y) = \lim_{n, m \rightarrow \infty} p(x_n, x_m).$$

Therefore, since $\{x_n\}$ is a Cauchy sequence in the metric space (Y, p^s) , it implies that $\lim_{n,m \rightarrow \infty} p^s(x_n, x_m) = 0$. Also from (2.3) we have $\lim_{n \rightarrow \infty} p(x_n, x_n) = 0$, and using the definition of p^s we obtain $\lim_{n,m \rightarrow \infty} p(x_n, x_m) = 0$. Consequently, from (2.4) we have

$$p(y, y) = \lim_{n \rightarrow \infty} p(x_n, y) = \lim_{n,m \rightarrow \infty} p(x_n, x_m) = 0.$$

As a result, $\{x_n\}$ is a Cauchy sequence in the complete partial metric subspace (Y, p) , and it is convergent to a point $y \in Y$.

On the other hand, the sequence $\{x_n\}$ has an infinite number of terms in each $A_i, i = 1, \dots, m$. Since (Y, p) is complete, in each $A_i, i = 1, \dots, m$, we can construct a subsequence of $\{x_n\}$ which converges to y . Since $A_i, i = 1, \dots, m$ are closed, we see that

$$\bigcap_{i=1}^m A_i; \text{ i.e.,}$$

$$\bigcap_{i=1}^m A_i \neq \emptyset. \text{ Now we can consider the restriction } f|_{\bigcap_{i=1}^m A_i} : \bigcap_{i=1}^m A_i \rightarrow \bigcap_{i=1}^m A_i,$$

which satisfies the conditions of Theorem 1 in [5, 6], since $\bigcap_{i=1}^m A_i$ is also closed and complete.

Thus $f|_{\bigcap_{i=1}^m A_i}$ has a unique fixed point, say $x^* \in \bigcap_{i=1}^m A_i$. We claim that for any initial value $x \in Y$, we get

the same limit point $x^* \in \bigcap_{i=1}^m A_i$. Indeed, for $x \in Y = \bigcup_{i=1}^m A_i$, by repeating the above process, the

corresponding iterative sequence yields that $f|_{\bigcap_{i=1}^m A_i}$ has a unique fixed point, say $z \in \bigcap_{i=1}^m A_i$. Regarding

that $x^*, z \in \bigcap_{i=1}^m A_i$, we have $x^* z \in A_i$ for all i , hence $p(x^*, z)$ and $p(f(x^*), f(z))$ are well defined. Due to (2.1),

we have

$$p(x^*, z) = p(f(x^*), f(z)) \leq \varphi(p(x^*, z)),$$

which is a contradiction. Thus, x^* is a unique fixed point of f for any initial value $x \in Y$.

To prove that the Picard iteration converges to x^* for any initial point $x \in Y$. Let $x \in Y = \bigcup_{i=1}^m A_i$. There

exists $i_0 \in \{1, \dots, m\}$ such that $x \in A_{i_0}$. As $x^* \in \bigcap_{i=1}^m A_i$ it follows that $x^* \in A_{i_0+1}$ as well. Then we obtain:

$$p(f(x), f(x^*)) \leq \varphi(p(x, x^*)).$$

By induction, it follows that: $p(f^n(x), x^*) \leq \varphi^n(p(x, x^*)), n \geq 0$.

Since $p(x^*, x^*) \leq p(f^n(x), x^*)$,

we have $p(x^*, x^*) \leq \varphi^n(p(x, x^*)).$

Now letting $n \rightarrow \infty$, and supposing $x \neq x^*$, we have

$$p(x^*, x^*) = \lim_{n \rightarrow \infty} p(f^n(x), x^*) = 0,$$

i.e., the Picard iteration converges to the unique fixed point of f for any initial point $x \in Y$.

Theorem 10 Let $f: Y \rightarrow Y$ as in Theorem 9. Then

$$\sum_{n=0}^{\infty} p(f^n(x), f^{n+1}(x)) < \infty, \text{ for any } x \in Y, \text{ i.e., } f \text{ is a good Picard operator.}$$

Proof. Let $x = x_0 \in Y$. Then $p(f^n(x_0), f^{n+1}(x_0)) = p(x^n, x^{n+1}) \leq \varphi^n(p(x_0, x_1))$.
for all $n \in \mathbb{N}$ Thus, by Lemma 6, we have

$$\sum_{n=0}^{\infty} p(f^n(x_0), f^{n+1}(x_0)) \leq \sum_{n=0}^{\infty} \varphi^n(p(x_0, x_1)) < \infty,$$

since $p(x_0, x_1) > 0$. So, f is a good Picard operator.

Theorem 11 Let $f: Y \rightarrow Y$ as in Theorem 9. Then $\sum_{n=0}^{\infty} p(f^n(x), x^*) < \infty$,

for any $x \in Y$, i.e., f is a special Picard operator.

Proof. Since $p(f^n(x), x^*) \leq \varphi^n(p(x, x^*))$, $n \geq 0$

holds for any $x \in Y$, by Lemma 6, we have $\sum_{n=0}^{\infty} p(f^n(x), x^*) \leq \sum_{n=0}^{\infty} \varphi^n(p(x_0, x_1)) < \infty$.

This shows that f is a special Picard operator.

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