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Laplace Substitution Method for nth Order Linear and Non-Linear PDE's Involving Mixed Partial Derivatives

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Abstract: *In this article, we will generalize the Laplace substitution method i.e we will give the description of Laplace substitution method for nth order linear and nonlinear partial differential equations involving mixed derivatives of three types* $\frac{\partial^n u}{\partial x^{n-1}\partial y}$, $\frac{\partial^n u}{\partial x\partial y^{n-1}}$, $\frac{\partial^n u}{\partial y^p\partial y^q}$ *(where p and q are positive integers such that p + q = n). This method we will demonstrated by four examples. The results obtained by the proposed method (LSM) tell us the method can be used for solution of the linear and nonlinear higher-order initial value problems involving mixed partial derivatives.*

Key Words: *Laplace transforms, Mixed partial derivatives, nth order partial derivatives, Laplace substitution method*

1. INTRODUCTION

It is well known that there are several methods that can be used to find general solutions to linear PDEs. On the contrary, for non-linear PDEs it is well known that there are no generally applicable methods to solve such nonlinear equations. A glance at the literature shows that there are some known methods which have been applied to solve special cases of nonlinear PDEs. For example the split-step method is a computational method that has been used to solve specific equations like nonlinear Schrödinger equation [5, 6]. Nevertheless, some techniques can be used to solve several types of nonlinear equations such as the homotopy principle which is the most powerful method to solve underdetermined equations [7]. In some cases, a PDE can be solved via perturbation analysis in which the solution is considered to be a correction to an equation with a known solution [8, 25]. Alternatively, there are numerical techniques that solve nonlinear PDEs such as the finite difference method [10, 11, and 12] and the finite element methods [13, 14 and 15]. We know that in [1][2][3][4], we have given the LSM for second order linear and nonlinear partial differential equations involving mixed partial derivatives.

In this article we will give the LSM for nth order linear and nonlinear partial differential equations involving mixed partial derivatives. The description of same method we will give in the next section (2), in this section we will give the three subsection, these three subsections are separated on the basis of types of mixed derivatives, namely $\frac{\partial^n u}{\partial x^{n-1}\partial y}$, $\frac{\partial^n u}{\partial x\partial y^{n-1}}$, $\frac{\partial^n u}{\partial y^p\partial y^q}$, (where p and q are positive integers such that p + q = n). After the description of (LSM), in section (3) we will solve this proposed method to four coupled partial differential equations involving mixed partial derivatives. In last section (4), we will give the conclusion of this article.

2. DESCRIPTION OF METHOD

In this section, we will give the only description of LSM for nth nonlinear partial differential equations involving mixed partial derivatives. If the equation is linear that means in the given equation does not contain nonlinear terms, and then we will use the same description for that linear equation, from that description we will remove only the part of use of Adomian polynomials. In the same section we will give the tree subsections, these are separated on the basis of types of mixed derivatives contained in the equation.

2.1 LSM for nth non linear partial differential equations involving mixed partial derivatives with mixed derivative of type $\frac{\partial^n u}{\partial x^{n-1}\partial y}$

Consider the following nth nonlinear partial differential equation,

$$Lu(x,y) + Nu(x,y) + Ru(x,y) = h(x,y) \qquad (2.1)$$

With initial conditions,

$u(x,0) = g(x), u_y(0,y) = f_0(y), u_{xy}(0,y) = f_1(y),$
$u_{x^2y}(0,y) = f_2(y), u_{x^3y}(0,y) = f_3(y), \dots\dots,$
$u_{x^{n-2}y}(0,y) = f_{n-2}(y)$ (2.2)

Where $L = \frac{\partial^n}{\partial x^{n-1}\partial y}$, Nu(x, y) is a nonlinear term,

Ru(x, y) is a remaining linear term and h(x, y) is a source term. Equation (2.1), we can write in the following form

$$\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial u}{\partial y}\right) + Nu(x,y) + Ru(x,y) = h(x,y)$$

Let we use the substitution, $U = \frac{\partial u}{\partial y}$ in above equation we get

$$\frac{\partial^{n-1}U}{\partial x^{n-1}} + Nu(x,y) + Ru(x,y) = h(x,y) \quad (2.3)$$

Taking Laplace transform of above equation with respect to x, we get

$$U(s,y) - \frac{1}{S}U(0,y) - \frac{1}{s^2}U_x(0,y) - \frac{1}{s^3}U_{x^2}(0,y)\dots\dots - \frac{1}{s^{n-1}}U_{x^{n-2}}(0,y) = \frac{1}{s^{n-1}}L_x[h(x,y) - Nu(x,y) - Ru(x,y)]$$

From the initial conditions (2.2) and the value of $U = \frac{\partial u}{\partial y}$, we get

$$U(S,y) - \frac{1}{S}f_0(y) - \frac{1}{s^2}f_1(y) - \frac{1}{s^3}f_2(y)\dots - \frac{1}{s^{n-1}}f_{n-2}(y) = \frac{1}{s^{n-1}}L_x[h(x,y) - Nu(x,y) - Ru(x,y)]$$

Taking inverse Laplace transform of above equation with respect to x, we get

$$U(x,y) = f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \dots + \frac{x^{n-3}}{(n-3)!}f_{n-3}(y) + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y) + L_x^{-1}\left[\frac{1}{s^{n-1}}L_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Re-substitute the value of $U = \frac{\partial u}{\partial y}$ in above equation, we get

$$\frac{\partial u}{\partial y} = f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \dots + \frac{x^{n-3}}{(n-3)!}f_{n-3}(y) + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y) + L_x^{-1}\left[\frac{1}{s^{n-1}}L_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Taking the Laplace transform of above equation with respect to y, we get

$$su(x,s) - u(x,0) = L_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \dots\dots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right] + L_y\left[L_x^{-1}\left[\frac{1}{s^{n-1}}L_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]$$

Again from the initial conditions (2.2), we get

$$u(x,s) = \frac{g(x)}{s} + \frac{1}{s}L_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \dots\dots\dots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right] + \frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^{n-1}}L_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]$$

Taking inverse Laplace transform of above equation with respect to y, we get

$$u(x,y) = g(x) + L_y^{-1}\left[\frac{1}{s}L_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \dots\dots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right]\right] + L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^{n-1}}L_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]\right] \quad (2.4)$$

Suppose that,

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \quad (2.5)$$

is a required solution of equation (2.1). The nonlinear term Nu(x, y) is appeared in equation (5.3.1). We can decompose it by using Adomian polynomial which is defined in [1, 2]

$$Nu(x,y) = \sum_{n=0}^{\infty} A_n \quad (2.6)$$

Where A_n is Adomian polynomial of components $u_0(x,y), u_1(x,y), \dots\dots.. u_n(x,y), n \geq 0$ of series (2.5). Substitute equations (2.5) and (2.6) in equation (2.4), we get

$$\sum_{n=0}^{\infty} u_n(x,y) = g(x) + L_y^{-1}\left[\frac{1}{s}L_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \dots\dots + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right]\right] + L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^{n-1}}L_x\left[h(x,y) - \sum_{n=0}^{\infty} A_n - R\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right]\right]\right]\right]$$

Comparing both sides of above equation, we get

$$u_0(x,y) = g(x) + L_y^{-1}\left[\frac{1}{s}L_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \dots\dots\frac{x^{n-3}}{(n-3)!}f_{n-2}(y) + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right]\right] + L_y^{-1}\left[\frac{1}{s}L_y\left[\frac{1}{s^{n-1}}L_x^{-1}[L_x[h]]\right]\right]$$

$$u_1(x,y) = -L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^{n-1}}L_x[A_0 + Ru_0(x,y)]\right]\right]\right]$$

$$u_2(x,y) = -L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^{n-1}}L_x[A_1 + Ru_1(x,y)]\right]\right]\right]$$

$$\vdots$$

$$u_n(x,y) = -L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^{n-1}}L_x[A_{n-1} + Ru_{n-1}(x,y)]\right]\right]\right], n \geq 1$$

From the above equations, we get the following recursive relation

$$u_0(x,y) = K(x,y)$$

$$u_{n+1}(x,y) = -L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^{n-1}}L_x[A_n + Ru_n(x,y)]\right]\right]\right], n \geq 0 \qquad (2.7)$$

Where

$$K(x,y) = g(x) + L_y^{-1}\left[\frac{1}{s}L_y\left[f_0(y) + xf_1(y) + \frac{x^2}{2!}f_2(y) + \dots + \frac{x^{n-3}}{(n-3)!}f_{n-2}(y) + \frac{x^{n-2}}{(n-2)!}f_{n-2}(y)\right]\right]$$
$$+ L_y^{-1}\left[\frac{1}{s}L_y\left[\frac{1}{s^{n-1}}L_x^{-1}[L_x[h]]\right]\right]$$

From the recursive relation (2.7), we can find the components $u_0(x,y), u_1(x,y), u_2(x,y), \dots u_n(x,y), n \geq 0$ of series (2.5). Substitute all these values in equation (2.5), we get the required solution of equation (2.1) in series form.

If in the (2.1) equation initial conditions are in the following form,

$$U_{x^{n-1}}(x,0) = g(x), u(0,y) = f_0(y),\ u_x(0,y) = f_1(y), u_{x^2}(0,y) = f_2(y),\ u_{x^3}(0,y) = f_3(y), \dots u_{x^{n-2}}(0,y) = f_{n-2}(y) \qquad (2.8)$$

then we can't use the substitution $U = \frac{\partial u}{\partial y}$ in equation (2.1). Hence we can't use the above explained method. Therefore we need to explain the LSM when such a situation appeared in given problems. By the Young's theorem, we know that the mixed derivative of n^{th} order $\frac{\partial^n u}{\partial x^{n-1}\partial y}$ appeared in equation (2.1), we can write it in the form of $\frac{\partial^n u}{\partial y\,\partial x^{n-1}}$. Therefore the n^{th} order nonlinear partial differential equation involving mixed derivatives (2.1) with initial conditions (2.8) becomes,

$$\frac{\partial^n u}{\partial y\,\partial x^{n-1}} + Nu(x,y) + Ru(x,y) = h(x,y) \qquad (2.9)$$

$$u_{x^{n-1}}(x,0) = g(x), u(0,y) = f_0(y),\ u_x(0,y) = f_1(y), u_{x^2}(0,y) = f_2(y),\ u_{x^3}(0,y) = f_3(y), \dots\dots\dots u_{x^{n-2}}(0,y) = f_{n-2}(y)$$

Above equation (2.9) we can write in the following form

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1}u}{\partial x^{n-1}}\right) + Nu(x,y) + Ru(x,y) = h(x,y)$$

Let we use the substitution $\frac{\partial^{n-1}u}{\partial x^{n-1}} = U$ in above equation, then above equation becomes

$$\frac{\partial U}{\partial y} + Nu(x,y) + Ru(x,y) = h(x,y)$$

Taking Laplace transform of above equation with respect to y, we get

$$U(x,s) = \frac{g(x)}{s} + \frac{1}{s}L_y[h(x,y) - Nu(x,y) - Ru(x,y)]$$

Taking inverse Laplace transform of above equation with respect to y, we get

$$U(x,y) = g(x) + L_y^{-1}\left[\frac{1}{s}L_y[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Re-substitute the value of U(x, y) in above equation, we get

$$\frac{\partial^{n-1}u(x,y)}{\partial x^{n-1}} = g(x) + L_y^{-1}\left[\frac{1}{s}L_y[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Taking Laplace transform of above equation with respect to x, we get

$$s^{n-1}u(s,y) - s^{n-2}u(0,y) - s^{n-3}u_x(0,y) - s^{n-4}u_{xx}(0,y) - \dots\dots\dots - u_{x^{n-2}}(0,y)$$
$$= L_x\left[g(x) + L_y^{-1}\left[\frac{1}{s}L_y[h(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]$$

From the initial conditions (2.8), we get

$$u(s,y) - \frac{f_0(y)}{s} - \frac{f_1(y)}{s^2} - \frac{f_2(y)}{s^3} - \dots - \frac{f_{n-2}(y)}{s^{n-1}}$$
$$= \frac{1}{s^{n-1}} L_x \left[g(x) + L_y^{-1}\left[\frac{1}{s} L_y[h(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]$$

Taking inverse Laplace transform of above equation with respect to x, we get

$$u(x,y) = f_0(y) + xf_1(y) + x^2\frac{f_2(y)}{2!} \dots + x^{n-2}\frac{f_{n-2}(y)}{(n-2)!}$$
$$+ L_x^{-1}\left[\frac{1}{s^{n-1}} L_x \left[g(x) + L_y^{-1}\left[\frac{1}{s} L_y[h(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]\right]$$

We know that in Laplace substitution method, we represent solution in infinite series form. Therefore suppose that,

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \qquad (2.10)$$

is a required solution of equation (2.9). As we have explained above at equation (2.6), similarly the nonlinear term Nu(x, y) we decompose it by using Adomian polynomial same as equation (2.6).
Substitute equation (2.10) and (2.6) in equation (2.9), we get

$$\sum_{n=0}^{\infty} u_n(x,y) = f_0(y) + xf_1(y) + x^2\frac{f_2(y)}{2!} + \dots + x^{n-2}\frac{f_{n-2}(y)}{(n-2)!} + L_x^{-1}\left[\frac{1}{s^{n-1}} L_x \left[g(x) + L_y^{-1}\left[\frac{1}{s} L_y[h(x,y) - \sum_{n=0}^{\infty} A_n - R[\sum_{n=0}^{\infty} u_n(x,y)]]\right]\right]\right] \qquad (2.11)$$

By comparing both sides of above equation, we get the following recursive relation

$$\left.\begin{aligned} u_0(x,y) &= k(x,y) \\ u_{n+1}(x,y) &= L_x^{-1}\left[\frac{1}{s^{n-1}} L_x\left[L_y^{-1}\left[\frac{1}{s} L_y[A_n + Ru_n(x,y)]\right]\right]\right] \end{aligned}\right\} \qquad (2.12)$$

Where

$$k(x,y) = f_0(y) + xf_1(y) + x^2\frac{f_2(y)}{2!} + \dots + x^{n-2}\frac{f_{n-2}(y)}{(n-2)!} + L_x^{-1}\left[\frac{1}{s^{n-1}} L_x[g(x)]\right] + L_x^{-1}\left[\frac{1}{s^{n-1}} L_x\left[L_y^{-1}\left[\frac{1}{s} L_y[h(x,y)]\right]\right]\right]$$

is a source term. From the recursive relation (2.12), we can find the components of the series (2.10). If we substitute the value of all these components in equation (2.10), we get the required solution of equation (2.9) with initial conditions (2.8).

2.2 LSM for n^{th} order nonlinear partial differential equations involving mixed partial derivatives with mixed derivative of type $\frac{\partial^n u}{\partial x \partial y^{n-1}}$

Consider the following n^{th} order nonlinear partial differential equation,

$$Lu(x,y) + Nu(x,y) + Ru(x,y) = h(x,y) \qquad (2.13)$$

In the above equation $L = \frac{\partial^n}{\partial x \partial y^{n-1}}$, Nu(x, y) is a nonlinear term, Ru(x, y) is a remaining linear term and h(x, y) is a source term. We can find the solution of equation (2.13) for the initial conditions,

I) $u_{y^{n-1}}(x,0) = g(y), u(x,0) = f_0(x), u_y(0,y) = f_1(y), u_y(0,y) = f_2(y), u_{y^3}(0,y) = f_3(y), \dots u_{y^{n-2}}(0,y) = f_{n-2}(y)$

(2.14)

II)
$u(x,0) = g(x), u_x(x,0) = f_0(x), u_{yx}(x,0) = f_1(x), u_{y^2x}(x,0) = f_2(x), u_{y^3x}(x,0) = f_3(x), \dots u_{y^{n-2}x}(x,0) = f_{n-2}(x)$

(2.15)

If the initial conditions are of **(I)**, then we will use the substitution $U = \frac{\partial^{n-1}}{\partial y^{n-1}}$, in equation (2.13). If the initial conditions are of the form **(II)**, then we will use the substitution $U = \frac{\partial u}{\partial x}$, in equation (2.13). As we have explained LSM for nonlinear partial differential equations involving mixed partial derivatives with mixed derivative of type $\frac{\partial^n u}{\partial x^{n-1} \partial y}$ in subsection (2.1), similarly we can find solution of nonlinear partial differential equations involving mixed partial derivatives with mixed derivative of type $\frac{\partial^n u}{\partial x \partial y^{n-1}}$ by LSM.

2.3 LSM for n^{th} order nonlinear partial differential equations involving mixed partial derivatives with mixed derivative of type $\frac{\partial^n u}{\partial x^p \partial y^q}$, where p and q are positive integers with p + q = n

Consider the following n^{th} order on linear partial differential equation,

$$Lu(x,y) + Nu(x,y) + Ru(x,y) = h(x,y) \qquad (2.16)$$

With initial conditions,

$$u_{y^q}(0,y) = f_0(y),\ u_{xy^q}(0,y) = f_1(y), u_{x^2y^q}(0,y) = f_2(y), \ldots\ldots\ldots u_{x^{p-2}y^q}(0,y) = f_{n-2}(y), u_{x^{p-1}y^q}(0,y)$$
$$f_{n-1}(y), u(x,0) = g_0(x),\ u_y(x,0) = g_1(x), u_{y^2}(x,0) g_2(x) \ldots - u_{y^{q-1}}(x,0) = g_{q-1}(x) \qquad (2.17)$$

In equation (2.16), $L = \frac{\partial^n}{\partial x^p \partial y^q}$, Ru(x, y) is a remaining linear term, Nu(x, y) is a nonlinear term and h(x, y) is a source term. Let we write the equation (2.16) in the following form,

$$\frac{\partial^n u}{\partial x^p\, \partial y^q} + Nu(x,y) + Ru(x,y) = h(x,y)$$

$$\frac{\partial^p}{\partial x^p}\left(\frac{\partial^q u}{\partial y^q}\right) + Nu(x,y) + Ru(x,y) = h(x,y)$$

Substitute $\frac{\partial^q u}{\partial y^q} = U$, in above equation we get

$$\frac{\partial^p U}{\partial x^p} + Nu(x,y) + Ru(x,y) = h(x,y)$$

Taking Laplace transform of above equation with respect to x, we get

$$U(s,y) - \frac{s^{p-1}}{s^p} U(0,y) - \frac{s^{p-2}}{s^p} U_x(0,y) - \frac{s^{p-3}}{s^p} U_{x^2}(0,y) \ldots\ldots - \frac{s}{s^p} U_{x^{p-2}}(0,y) - \frac{1}{s^p} U_{x^{p-1}}(0,y) = \frac{1}{s^p} L_x[h(x,y) - Nu(x,y) - Ru(x,y)]$$

By using the value of $U = \frac{\partial^q u}{\partial y^q}$ in above equation, we get

$$U(s,y) - \frac{1}{S} u_{y^q}(0,y) - \frac{1}{s^2} u_{xy^q}(0,y) - \frac{1}{s^3} u_{x^2y^q}(0,y) - \ldots\ldots\ldots - \frac{1}{s^p} u_{x^{p-1}y^q}(0,y) = = \frac{1}{s^p} L_x[h(x,y) - Nu(x,y) - Ru(x,y)]$$

From the initial conditions (5.3.25) and above equation, we get

$$U(S,y) - \frac{1}{S} f_0(y) - \frac{1}{s^2} f_1(y) - \frac{1}{s^3} f_2(y) - \cdots - \frac{1}{s^{p-1}} f_{p-2}(y) - \frac{1}{s^p} f_{p-1}(y) = \frac{1}{s^p} L_x[h(x,y) - Nu(x,y) - Ru(x,y)]$$

Taking inverse Laplace transform of above equation with respect to x, we get

$$U(x,y) = f_0(y) + x f_1(y) + \frac{x^2}{2!} f_2(y) + \cdots + \frac{x^{p-2}}{(p-2)!} f_{p-2}(y) + \frac{x^{p-1}}{(p-1)!} f_{p-1}(y) + L_x^{-1}\left[\frac{1}{s^p} L_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Let we use the value of $U = \frac{\partial^q u}{\partial y^q}$, in above equation, we get

$$\frac{\partial^q u}{\partial y^q} = f_0(y) + x f_1(y) + \frac{x^2}{2!} f_2(y) \ldots. + \frac{x^{p-1}}{(p-1)!} f_{p-1}(y) + L_x^{-1}\left[\frac{1}{s^p} L_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]$$

Taking the Laplace transform of above equation with respect to y, we get

$$u(x,s) = \frac{1}{s} g_0(x) + \frac{1}{s^2} g_1(x) + \frac{1}{s^3} g_2(x) \ldots + \frac{1}{s^q} g_{q-1}(x) + \frac{1}{s^q} L_y\left[f_0(y) + x f_1(y) + \frac{x^2}{2!} f_2(y) \ldots + \frac{x^{p-2}}{(p-2)!} f_{p-2}(y) + \frac{x^{p-1}}{(p-1)!} f_{p-1}\right] + \frac{1}{s^q} L_y\left[L_x^{-1}\left[\frac{1}{s^p} L_x[h(x,y) - Nu(x,y) - Ru(x,y)]\right]\right]$$

Taking inverse Laplace transform of above equation with respect to y, we get

$$u(x,y) = g_0(x) + y g_1(x) + \frac{y^2}{2!} g_2(x) \ldots + \frac{y^{q-1}}{(q-1)!} g_{q-1}(x) + L_y^{-1}\left[\frac{1}{s^q} L_y\left[f_0(y) + x f_1(y) + \frac{x^2}{2!} f_2(y) \ldots + \right.\right.$$

$$\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}\Big]\Big]+L_y^{-1}\left[\frac{1}{s^q}L_y\left[L_x^{-1}\left[\frac{1}{s^p}L_x[h(x,y)-Nu(x,y)-Ru(x,y)]\right]\right]\right] \quad (2.18)$$

We know that, in Laplace Substitution Method we represent solution in infinite series form; therefore let we suppose that

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y) \quad (2.19)$$

is a required solution of equation (2.16). Let we decompose nonlinear term Nu(x, y) by using Adomian polynomial A_n.

$$Nu(x,y)=\sum_{n=0}^{\infty}A_n \quad (2.20)$$

Substitute equation (2.19) and (2.20) in (2.18), we get

$$\sum_{n=0}^{\infty}u_n(x,y)=g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)\dots+\frac{y^{q-2}}{(q-2)!}g_{q-2}(x)+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)+L_y^{-1}\left[\frac{1}{s^q}L_y\left[f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)\dots+\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}\right]\right]+L_y^{-1}\left[\frac{1}{s^q}L_y\left[L_x^{-1}\left[\frac{1}{s^p}L_x\left[h(x,y)-\sum_{n=0}^{\infty}A_n-R\left(\sum_{n=0}^{\infty}u_n(x,y)\right)\right]\right]\right]\right]$$

Comparing both sides of above equation, we get the following recursive relation

$$\left.\begin{aligned}u_0(x,y)&=K(x,y)\\ u_{n+1}(x,y)&=L_y^{-1}\left[\frac{1}{s^q}L_y\left[L_x^{-1}\left[\frac{1}{s^p}L_x[A_{n-1}+Ru_n(x,y)]\right]\right]\right]\end{aligned}\right\} \quad (2.21)$$

Where

$$K(x,y)=g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)\dots+\frac{y^{q-2}}{(q-2)!}g_{q-2}(x)+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)+L_y^{-1}\left[\frac{1}{s^q}L_y\left[f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)\dots+\frac{x^{p-2}}{(p-2)!}f_{p-2}(y)+\frac{x^{p-1}}{(p-1)!}f_{p-1}\right]\right]+L_y^{-1}\left[\frac{1}{s^q}L_y\left[L_x^{-1}\left[\frac{1}{s^p}L_x[h(x,y)]\right]\right]\right]$$

From the recursive relation (2.21), we can find the components $u_0(x,y),u_1(x,y),u_2(x,y),\dots\dots\dots\dots\dots u_n(x,y),n\geq 0$ of the series (2.19). Substitute all these values in equation (2.19), we get the required solution of equation (2.16) with initial conditions (2.17).

Let we solve the nonlinear equation (2.16) with following initial conditions

$$u_{x^p}(x,0)=g_0(x),\ u_{yx^q}(x,0)=g_1(x),u_{y^2x^q}(x,0)=g_2(x),\dots,u_{y^{q-2}x^p}(x,0)=g_{q-2}(x),$$
$$u_{y^{q-1}x^p}(x,0)=g_{q-1}(x),u(0,y)=f_0(y),$$
$$u_x(0,y)=f_1(y),u_{x^2}(0,y)=f_2(y),\dots\ u_{x^{p-1}}(0,y)=f_{p-1}(y)$$
(2.22)

then by the Young's theorem, equation (2.16) we can write in the following form

$$\frac{\partial^n u}{\partial y^q\partial x^p}+Nu(x,y)+Ru(x,y)=h(x,y)$$

$$\frac{\partial^q}{\partial y^q}\left(\frac{\partial^p u}{\partial x^p}\right)+Nu(x,y)+Ru(x,y)=h(x,y)$$

Let we use the substitute $\frac{\partial^p u}{\partial x^p}=U$, in above equation we get

$$\frac{\partial^q U}{\partial y^q}+Nu(x,y)+Ru(x,y)=h(x,y)$$

Taking Laplace transforms and then inverse Laplace transform of above equation with respect to y, we get

$$U(x,y)=g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)+\cdots+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)$$

$$+L_y^{-1}\left[\frac{1}{s^q}L_y[h(x,y)-Nu(x,y)-Ru(x,y)]\right]$$

Let we use the value $U=\frac{\partial^p u}{\partial x^p}$, in above equation, we get

$$\frac{\partial^p u}{\partial x^p}=g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)\ldots+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)$$
$$+L_y^{-1}\left[\frac{1}{s^q}L_y[h(x,y)-Nu(x,y)-Ru(x,y)]\right]$$

Taking the Laplace transform and then inverse Laplace transform of above equation with respect to x, we get

$$u(x,y)=f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)..+\frac{x^{q-1}}{(q-1)!}f_{p-1}(y)$$
$$+L_x^{-1}\left[\frac{1}{s^p}L_x\left[g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)\ldots+\frac{y^{q-2}}{(q-2)!}g_{q-2}(x)+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)\right]\right]$$
$$+L_x^{-1}\left[\frac{1}{s^p}L_x\left[L_y^{-1}\left[\frac{1}{s^q}L_y[h(x,y)-Ru(x,y)]\right]\right]\right] \quad (2.23)$$

Let we suppose that,

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y) \quad (2.24)$$

be a required solution of equation (2.16) with initial conditions (2.22).

We have decomposed nonlinear term Nu(x, y) by equation (2.20), similarly we can decompose it by using Adomian polynomial A_n same way. Substitute equation (2.24) and (2.20) in (2.23), we get

$$\sum_{n=0}^{\infty}u_n(x,y)=f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)\ldots\ldots$$
$$+\frac{x^{q-1}}{(q-1)!}f_{p-1}(y)$$
$$+L_x^{-1}\left[\frac{1}{s^p}L_x\left[g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)\ldots+\frac{y^{q-2}}{(q-2)!}g_{q-2}(x)+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)\right]\right]+L_x^{-1}\left[\frac{1}{s^p}L_x\left[L_y^{-1}\left[\frac{1}{s^q}L_y[h(x,y)-\sum_{n=0}^{\infty}A_n-R(\sum_{n=0}^{\infty}u_n(x,y))]\right]\right]\right]$$

(2.25)

Comparing both sides of above equation, we get the following recursive relation

$$\left.\begin{aligned}&u_0(x,y)=K(x,y)\\&u_{n+1}(x,y)=-L_x^{-1}\left[\frac{1}{s^p}L_x\left[L_y^{-1}\left[\frac{1}{s^q}L_y[A_n+Ru_n(x,y)]\right]\right]\right]\end{aligned}\right\} \quad (2.26)$$

Where,

$$K(x,y)=f_0(y)+xf_1(y)+\frac{x^2}{2!}f_2(y)\ldots\ldots+\frac{x^{q-1}}{(q-1)!}f_{p-1}(y)$$
$$+L_x^{-1}\left[\frac{1}{s^p}L_x\left[g_0(x)+yg_1(x)+\frac{y^2}{2!}g_2(x)\ldots+\frac{y^{q-2}}{(q-2)!}g_{q-2}(x)+\frac{y^{q-1}}{(q-1)!}g_{q-1}(x)\right]\right]+L_x^{-1}\left[\frac{1}{s^p}L_x\left[L_y^{-1}\left[\frac{1}{s^q}L_y[h(x,y)]\right]\right]\right]$$

From the recursive relation (2.26), we can find $u_0(x,y),u_1(x,y),u_2(x,y),\ldots\ldots\ldots\ldots\ldots u_n(x,y),n\geq 0$. Substitute all these values in equation (2.24), we get the required solution of equation (2.16) with initial conditions (2.22).

3. APPLICATION OF LSM FOR n^{th} ORDER PDE'S INVOVLING MIXED PARTIAL DERIVATIVES

In this section we will solve the four partial differential equations involving mixed partial derivatives with given initial conditions.

Example 1: Let we consider the first following linear partial differential equation of 5^{th}order

$$\frac{\partial^5 u}{\partial x\partial y^4}+\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=y-x \quad (3.1)$$

With initial conditions

$$\left.\begin{aligned}u(x,0)=0,u(0,y)=0,u_y(x,0)=x,u_{yy}(x,0)=0\\u_{yyy}(x,0)=0,u_{yyyy}(0,y)=0\end{aligned}\right\} \quad (3.2)$$

Equation (3.1) we can write in the following form

$$\frac{\partial}{\partial x}\left(\frac{\partial^4 u}{\partial y^4}\right)+\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=y-x$$

Because of the given initial condition, let we use the substitution $U=\frac{\partial^4 u}{\partial y^4}$ in above equation, we get

$$\frac{\partial U}{\partial x}+\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=y-x$$

Taking the Laplace transform of above equation with respect to x, and then inverse Laplace transform of above equation, we get

$$\frac{\partial^4 u}{\partial y^4}=yx-\frac{x^2}{2!}-u(x,y)+L_x^{-1}\left[\frac{1}{s}L_x\left[\frac{\partial u}{\partial y}\right]\right]$$

Taking Laplace transform of above equation with respect y, then inverse Laplace transform of above equation, we get

$$u(x,y)=yx+\frac{xy^5}{5!}-\frac{x^2y^4}{2!\,4!}-L_y^{-1}\left[\frac{1}{s^4}L_y[u(x,y)]\right]$$
$$+L_y^{-1}\left[\frac{1}{s^4}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x\left[\frac{\partial u}{\partial y}\right]\right]\right]\right] \qquad (3.3)$$

Suppose that,

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y) \qquad (3.4)$$

is a required solution of equation of equation (3.1) with initial conditions (3.2).We are not using Adomain polynomial here, because of absence of nonlinear term. Substitute equation (3.4) in equation (3.3), we get

$$\sum_{n=0}^{\infty}u_n(x,y)=yx+\frac{xy^5}{5!}-\frac{x^2y^4}{2!\,4!}-L_y^{-1}\left[\frac{1}{s^4}L_y\left[\sum_{n=0}^{\infty}u_n(x,y)\right]\right]+L_y^{-1}\left[\frac{1}{s^4}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x\left[\sum_{n=0}^{\infty}u_{ny}(x,y)\right]\right]\right]\right]$$

Let we use here Modified Laplace Substitution Method [2, 3, 4], comparing both sides of above equation, we get the following recursive relation,

$$u_0(x,y)=yx$$

$$u_1(x,y)=\frac{xy^5}{5!}-\frac{x^2y^4}{2!\,4!}-L_y^{-1}\left[\frac{1}{s^4}L_y[u_0]\right]+L_y^{-1}\left[\frac{1}{s^4}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x[u_{0y}]\right]\right]\right]$$

$$u_n(x,y)=-L_y^{-1}\left[\frac{1}{s^4}L_y[u_{n-1}]\right]+L_y^{-1}\left[\frac{1}{s^4}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x[u_{(n-1)y}]\right]\right]\right] \qquad (3.5)$$

From the above recursive relation, let we calculate components of series solution (3.4)

$$u_0(x,y)=yx$$

$$u_1(x,y)=\frac{xy^5}{5!}-\frac{x^2y^4}{2!\,4!}-L_y^{-1}\left[\frac{1}{s^4}L_y[u_0(x,y)]\right]+L_y^{-1}\left[\frac{1}{s^4}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x[u_{0y}(x,y)]\right]\right]\right]=-\frac{x^2y^4}{2!\,4!}+\frac{x^2y^4}{2!\,4!}=0$$

Similarly,

$$u_2(x,y)=-L_y^{-1}\left[\frac{1}{s^4}L_y[u_1(x,y)]\right]$$
$$+L_y^{-1}\left[\frac{1}{s^4}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x[u_{1y}(x,y)]\right]\right]\right]=0 \quad \text{........ so on}$$

Substitute all the values of $u_n(x,y)$, $n \geq 0$, in equation (3.4), we get

$$u(x,y)=yx$$

This is an exact solution of equation (3.1) with initial conditions (3.2). We have verified this through the substitution.

Example 2: Consider the following 3rd order nonlinear partial differential equation

$$\frac{\partial^3 u}{\partial x^2\partial y}-2\left(\frac{\partial u}{\partial y}\right)^3=0 \qquad (3.6)$$

with initial condition

$$u_y(0,y)=1,\ u_{xy}(0,y)=1,\ u(x,0)=0 \qquad (3.7)$$

Let we use the substitution $U=\frac{\partial u}{\partial y}$ above equation, we get

$$\frac{\partial^2 U}{\partial x^2}-2U^3=0$$

Taking Laplace transform of above equation with respect to x, and then taking inverse Laplace transform of above equation, we get

$$U(x,y) = 1 + x + L_x^{-1}\left[\frac{1}{s^2}L_x[2U^3]\right]$$

Re-substitute the value of $U = \frac{\partial u}{\partial y}$, in above equation, we get

$$\frac{\partial u}{\partial y} = 1 + x + L_x^{-1}\left[\frac{1}{s^2}L_x\left[2\left(\frac{\partial u}{\partial y}\right)^3\right]\right]$$

Taking Laplace transform and then inverse Laplace Transform of above equation with respect to y, we get

$$u(x,y) = y(1+x) + 2L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^2}L_x\left[\left(\frac{\partial u}{\partial y}\right)^3\right]\right]\right]\right] \quad (3.8)$$

Suppose that,

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \quad (3.9)$$

is a required solution of equation (3.6). We know that a nonlinear term $Nu(x,y) = \left(\frac{\partial u}{\partial y}\right)^3$ is appear in given equation (3.6), let we decompose it by using Adomian polynomial which is defined [2, 3, 4]

$$\left(\frac{\partial u}{\partial y}\right)^3 = \sum_{n=0}^{\infty} A_n \quad (3.10)$$

In above equation $A_n, n \geq 0$ is an Adomian polynomial of components $u_0(x,y), u_1(x,y), u_2(x,y), \ldots\ldots\ldots, u_n(x,y),\ n \geq 0$ of series (3.9). We have found the some Adomian polynomials A_n, by using the formula [2, 3, 4].

$$A_0 = u_{0y}^3,\ A_1 = 3u_{0y}^2\,u_{1y},\ A_2 = 3u_{0y}u_{1y}^2 + 3u_{0y}^2\,u_{2y},\ \ldots\ldots \text{ so on}$$

Substitute equations (3.9) and (3.10) in (3.8), we get

$$\sum_{n=0}^{\infty} u_n(x,y) = y(1+x) + 2L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^2}L_x\left[\sum_{n=0}^{\infty} A_n\right]\right]\right]\right]$$

Let we use again MLSM here for finding exact solution of equation (3.6) with less computation. Comparing both sides of above equation, we get the following recursive relation

$$\left.\begin{aligned} u_0(x,y) &= y \\ u_1(x,y) &= xy + 2L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^2}L_x[A_0]\right]\right]\right] \\ u_{n+1}(x,y) &= 2L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^2}L_x[A_n]\right]\right]\right], n > 1 \end{aligned}\right\} \quad (3.11)$$

From the above recursive relation, let we calculate the components $u_n(x,y), n \geq 0$.

$$u_0(x,y) = y$$

$$u_1(x,y) = xy + 2L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^2}L_x[A_0]\right]\right]\right] = xy + x^2y,$$

$$u_2(x,y) = 2L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^2}L_x[A_1]\right]\right]\right] = y\left[x^3 + \frac{x^4}{2}\right],$$

$$u_3(x,y) = 2L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s^2}L_x\left[3u_{0y}u_{1y}^2 + 3u_{0y}^2\,u_{2y}\right]\right]\right]\right] = y\left[\frac{3x^5}{2} + \frac{x^4}{2} + \frac{x^6}{2}\right],$$

……… so on.

Substitute all the above values in equation (3.9), we get

$$u(x,y) = y(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots\ldots) = y\sum_{n=0}^{\infty} x^n$$

$\sum_{n=0}^{\infty} x^n$ is a geometric series converges to $\frac{1}{1-x}$, for $|x| < 1$. Therefore,

$$u(x,y) = \frac{y}{1-x}, |x| < 1$$

This is an exact solution of equation (3.6) with initial conditions (3.7). We have verified this through the substitution.

Example 3: Consider the following 5th order nonlinear partial differential equation

$$\frac{\partial^5 u}{\partial x^3 \partial y^2} - u\frac{\partial u}{\partial x} + u^2 = e^x \cos y \quad (3.12)$$

with initial conditions

$$u(0,y) = -\cos y, u_x(0,y) = -\cos y,\ u_{x^2}(0,y) = -\cos y, u_{x^3}(x,0) = -e^x, u_{x^3y}(x,0) = 0 \quad (3.13)$$

Because of the initial conditions (3.6) we cannot use the substitution $U = \frac{\partial^2 u}{\partial y^2}$ in above equation (3.12). In this problem we can use the substitution $U = \frac{\partial^3 u}{\partial x^3}$. Therefore let we write the equation (3.12) in following form

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^3 u}{\partial x^3}\right) - u\frac{\partial u}{\partial x} + u^2 = e^x \cos y$$

Let we use the substitution $U = \frac{\partial^3 u}{\partial x^3}$ in above equation, we get

$$\frac{\partial^2 U}{\partial y^2} - u\frac{\partial u}{\partial x} + u^2 = e^x \cos y$$

Taking Laplace transform and then inverse Laplace transform of above equation with respect to y, we get

$$U(x,y) + e^x - L_y^{-1}\left[\frac{1}{s^2}L_y\left[u\frac{\partial u}{\partial x} - u^2\right]\right] = e^x[1-\cos y]$$

Re-substitute the value of $U = \frac{\partial^3 u}{\partial x^3}$, in above equation, we get

$$\frac{\partial^3 u}{\partial x^3} = -e^x \cos y + L_y^{-1}\left[\frac{1}{s^2}L_y\left[u\frac{\partial u}{\partial x} - u^2\right]\right]$$

Taking Laplace transform then inverse Laplace transform of above equation with respect to x, we get

$$u(x,y) = -e^x \cos y + L_x^{-1}\left[\frac{1}{s^3}L_x\left[L_y^{-1}\left[\frac{1}{s^2}L_y\left[u\frac{\partial u}{\partial x} - u^2\right]\right]\right]\right]$$

(3.14)

Suppose that,

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \qquad (3.15)$$

is a required solution of equation (3.6). We know that a nonlinear term $Nu(x,y) = u\frac{\partial u}{\partial x} - u^2$ is appear in given equation (3.6), let we decompose it by using Adomian polynomial which is defined [2, 3, 4]

$$u\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} A_n \quad \text{and} \quad u^2 = \sum_{n=0}^{\infty} B_n \qquad (3.16)$$

In above equation A_n and B_n, $n \geq 0$ are Adomian polynomials of components $u_0(x,y), u_1(x,y), u_2(x,y), \ldots\ldots\ldots\ldots, u_n(x,y)$, $n \geq 0$ of series (3.15). We have found the some Adomian polynomials A_n and B_n ,

$$A_0 = u_0 u_{0x}\ ,\ B_0 = u_0^2$$

$$A_1 = u_0 u_{1x} + u_1 u_{0x}\ ,\ B_1 = 2u_0 u_1$$

$$A_2 = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}\ ,\ B_2 = 2u_0 u_2 + u_1^2$$

……………………. so on.

Substitute equations (3.15) and (3.16) in (3.14), we get

$$\sum_{n=0}^{\infty} u_n(x,y) = -e^x \cos y + L_x^{-1}\left[\frac{1}{s^3}L_x\left[L_y^{-1}\left[\frac{1}{s^2}L_y\left[\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n\right]\right]\right]\right]$$

Comparing both sides of above equation, we get the following recursive relation

$$\left.\begin{aligned} u_0(x,y) &= -e^x \cos y \\ u_n(x,y) &= L_x^{-1}\left[\frac{1}{s^3}L_x\left[L_y^{-1}\left[\frac{1}{s^2}L_y[A_{n-1} - B_{n-1}]\right]\right]\right], n > 0 \end{aligned}\right\} \qquad (3.17)$$

From the above recursive relation, let we calculate the components $u_n(x,y), n \geq 0$.

$$u_0(x,y) = -e^x \cos y$$

$$u_1(x,y) = L_x^{-1}\left[\frac{1}{s^3}L_x\left[L_y^{-1}\left[\frac{1}{s^2}L_y[A_0 - B_0]\right]\right]\right] = 0$$

$$\vdots$$

$$u_n(x,y) = 0, n \geq 1$$

Substitute all these values in equation (3.15), we get

$$u(x,y) = -e^x \cos y$$

This is an exact solution of equation (3.6) with initial conditions (3.7). We have verified this through the substitution.

4. CONCLUSION

The proposed Laplace substitution method has been successfully applied directly to n^{th} order linear and nonlinear PDEs involving any type of mixed partial derivatives without using linearization, perturbation, or restrictive assumptions. Moreover from examples (1) and (2), we can say that MLSM is easily applicable to same problems. It provides the solution in terms of convergent series with easily computable components and the results have shown remarkable performance. The efficiency of this method has been demonstrated by solving one linear and two nonlinear PDEs of any order.

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