ON THE DIFFERENTIABILITY OF FUZZY TRANSITION PROBABILITY OF FUZZY MARKOV CHAINS

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ABSTRACT: *In this paper the notion of fuzzy logic is mixed with the theory of Markov systems and the concept of a Markov system with fuzzy states is introduced. The differentiability of fuzzy transition probability and Markov fuzzy process with a transition probability measures and a general state space are attempted. Here we defined the probability of the fuzzy event and the α-level set of fuzzy transition function. The Axiomatic Definition of fuzzy transition probability P_{FRFS} and the results pertaining to it. We also provide the Kolmogorovs Differential differences equation.*

KEYWORDS: *Fuzzy Markov chain, Fuzzy probability and Fuzzy transition probability*

1. INTRODUCTION

The fuzzy Markov Chains approaches are given by Avarchenkov and Sanchez [1] by using max-min operator on fuzzy sets to determine their stationary behavior. Smith [4] describes the set of stationary probability vectorU arising when the transition probabilities of an n-state Markov chain vary over a specified range. Takahashi [5] obtained simple bands on the limiting state vector and other characteristics of a finite Markov chain whose P_{ij} values vary over a specified range. In the same line an attempt to relax the restriction was proposed by Skulj [3] where the assumption of precisely known initial and Transition probabilities is relaxed so that probability intervals are used instead of precise probabilities.

Let us now provide the prime motivating factors for considering fuzzy sets in Markov systems. In some situations in real applications one is often faced it the fact of fuzzy states in the sense that the states of the system cannot be precisely measured and they are bound to be fuzzy. These sets are perceived as having imprecise boundaries that facilitate gradual transition from membership to non-membership and vice versa. This property provides not only a meaningful and powerful representation of measurement uncertainties but also a meaningful representation of vague concepts expressed in natural language.

A Markov system can be used to describe a dynamic system that evolves over time according to probability laws and it can be applied to problems of different scientific areas such as Ecology, Sociology, Human Geography management etc.

Moreover in [2] an interesting use of Transition matrices in certain ecological population is discussed. In [7] Yoshida constructed a Markov fuzzy process with a transition possibility measure and a general state space.

In section 2 we briefly state what we mean by a fuzzy Markov chain. In section 2 we have defined the probability of the fuzzy event and the α-level set of fuzzy transition function.

In Section 3 we provide the Axiomatic Definition of fuzzy transition probability P_{FRFS} and the results pertaining to it.

Section 4 is devoted to fuzzy transition function and the limit properties of fuzzy transition probability.

Section 5 deals with the Kolmogorov differential difference equation.

1. FUZZY TRANSITION PROBABILITIES:

Let S={1,2,.....,k} be the state space. Let F= $\{F_1,F_2,\ldots\ldots.F_N\}$ be the fuzzy state space. The set of Fuzzy states Fr is assumed to be a fuzzy set on s and $\mu_{F_r}(.): S \to [0,1]$ denote the membership function of the fuzzy set Fr, for r=1,2,.....,N. It is also assumed that F= $\{F_1,F_2,\ldots\ldots.F_N\}$ defines a fuzzy partition on S such that $\sum_{r=1}^{N} \mu_{F_r}(.) = 1$. This is the fuzzy probabilistic pattern of S.

2.1 DEFINITION:

[1] Let (Ω, A, P) be the standard probability space where Ω denotes the sample space. A the σ-algebra on Ω and P the probability measure. A fuzzy set on Ω is called a fuzzy event. Let $\mu_A(.)$ be the membership function of a fuzzy set A. Then the probability of the fuzzy event A is defined by

$$P(A) = \int_{\Omega} \mu_A(w)dp \; ; \mu_A(w): \Omega \to [0,1]$$

Unless otherwise specifically stated let F_r denote the initial state, F_s denote the terminal state and F_k denote the intermediate state.

2.2 DEFINITION:

The Markov chain {Xn; n ≥ 0} is completely determined by the transition matrix P.

$P(X_{n+1} = F_s | X_n = F_r) = P_{F_rF_s}$

2.3 DEFINITION:

$P_{F_rF_s}$ is the fuzzy transition probability if and only if $(P_{F_rF_s})_\alpha = [(P_{F_rF_s})_\alpha^-, (P_{F_rF_s})_\alpha^+]$ is a transition probability for every $\alpha \in [0,1]$ and $P_{F_rF_s}(+) = \cup_{\alpha \in [0,1]} \alpha (P_{F_rF_s})_\alpha (+)$

2. AXIOMATIC DEFINITION

Here T={t; 0≤t≤∞}, S= denumerable set. The Markov chain transition probabilities satisfy the following conditions.

1) $0 \leq P_{FrFs} \leq 1$
2) $\sum_{Fs} P_{FrFs} = 1$
3) $P_{FrFs}^{n+1} = \sum_{Fk\in s} P_{FrFk}^{(n)} P_{FrFs}$
4) $P_{FrFs}^{(0)} = \delta_{FrFs}$ (*Kromecker delta*)

Now for the fuzzy states

$P_{FrFs}^{(t)} = P[X(t) = Fs | X(0) = Fr]$

$= P[X(t+s) = Fs | X(s) = Fr]$

As the transition function of a continuous parameter Markov Process. Let {X(t); t∈R⁺} be a Markov process with countable state space S and transition probability function $P_{FrFs}(t)$, Fr,Fs∈S; $t \in R^+$.

We assume that following axioms about transition function $P_{FrFs}(t)$

1. $0 \leq P_{FrFs}(t) \leq t$
2. $\sum_{Fs} P_{FrFs}(t) = 1 \quad t \geq 0$
3. *Champman – Kolmagorov equation*

$P_{FrFs}{}^{(t+h)} = \sum_{Fk\in s} P_{FrFk}{}^{(t)} P_{FkFs}{}^{(h)}; \quad t \geq 0; h \geq 0$

4. $P_{FrFs}{}^{(0)} = \delta_{FrFs}$
5. $\lim_{t\downarrow 0} P_{FrFs}{}^{(t)} = \delta_{FrFs}$

Note that the above axioms (1) to (5) are right continuous function at t=0. It can also be written as 1≥ p(t) ≥ 0 where 1 and 0 are elements with 1 and 0 respectively.

LEMMA:3.1

P(t) is continuous for $t \geq 0$.

Proof:

From Chapman Kolmogorov equation

$P_{FrFs}{}^{(t+h)} = \sum_{Fk\in s} P_{FrFk}{}^{(t)} P_{FkFs}{}^{(h)} \quad ; \quad t \geq 0; h \geq 0$

P(t+s) = P(t) P(S) ; $\quad t \geq 0; S \geq 0$

(3.1)

If S=h>0 then we have

$\lim_{h\to 0+} P(t+h) = \bigcup_{\alpha\in(0,1]} \lim_{h\to 0+} \alpha P_\alpha(t+h)$

$= \bigcup_{\alpha\in(0,1]} \alpha P_\alpha(t) \lim_{h\to 0+} P(h)$

$= \bigcup_{\alpha\in(0,1]} \alpha P_\alpha(t) I$

$= \bigcup_{\alpha\in(0,1]} \alpha P_\alpha(t)$

$= P(t)$

$i.e$ $P(t)$ is right continuous.

$If\ t > 0\ and\ 0 < h < t$

P(t)=P(t-h)P(h)
(3.2)

$\bigcup_{\alpha\in(0,1]} \alpha P_\alpha(t) = \bigcup_{\alpha\in(0,1]} \alpha P_\alpha(t-h) P_\alpha(h)$

But | $P_\alpha(h)$ - I |< ∈ if |h| <δ (by axiom 5) and for every $\alpha \in (0,1]$

Hence for |h| < δ, $P_\alpha^{-1}(h)$ exists and tends to I as h→0

Therefore $P(\alpha) = \bigcup_{\alpha\in(0,1]} \alpha P_\alpha(t)$

$= \bigcup_{\alpha\in(0,1]} \alpha P_\alpha(t) \lim_{h\to 0+} P_\alpha^{-1}(h)$

$= \bigcup_{\alpha\in(0,1]} \alpha \lim_{h\to 0+} (P_\alpha(t)\, P_\alpha^{-1}(h))$

$= \bigcup_{\alpha\in(0,1]} \alpha \lim_{h\to 0+} P_\alpha(t-h)\, P_\alpha(\mathrm{h}) P_\alpha^{-1}(h)$

$= \bigcup_{\alpha\in(0,1]} \alpha \lim_{h\to 0+} P_\alpha(t-h)$

$= \lim_{h\to 0+} P\ (t-h)$

Thus P(t) is left continuous and hence P(t) is continuous.

[From Champman Kolmogorov equation

P(t+s) = P(t) P(S) ; $t \geq 0; S \geq 0$

i.e. $\bigcup_{\alpha\in(0,1]} \alpha P_\alpha(t+s) = \bigcup_{\alpha\in(0,1]} \alpha P_\alpha(t) P_\alpha(s)$

$\bigcup_{\alpha\in(0,1]} \alpha[P_\alpha(t+s) - P_\alpha(t)] = \bigcup_{\alpha\in(0,1]} \alpha[P_\alpha(t)P_\alpha(s) - P_\alpha(t)]$
$= \bigcup_{\alpha\in(0,1]} \alpha[P_\alpha(t)P_\alpha(s) - I]$

$\lim_{s\downarrow 0} \bigcup_{\alpha\in(0,1]} \alpha \frac{[P_\alpha(t+s)-P_\alpha(t)]}{s} = \lim_{s\downarrow 0} \bigcup_{\alpha\in(0,1]} \alpha \frac{[P_\alpha(t)P_\alpha(s)-I]}{s}$

$\bigcup_{\alpha\in(0,1]} \alpha \lim_{s\downarrow 0} \frac{[P_\alpha(t+s)-P_\alpha(t)]}{s} = \bigcup_{\alpha\in(0,1]} \alpha \lim_{s\downarrow 0} \frac{[P_\alpha(t)P_\alpha(s)-I]}{s}$

$\bigcup_{\alpha\in(0,1]} \alpha P'_\alpha(t) = \bigcup_{\alpha\in(0,1]} \alpha\ P_\alpha(t) \lim_{s\downarrow 0} \frac{[P_\alpha(s)-I]}{s}$

If $\lim_{s\downarrow 0} \frac{[P_\alpha(s)-I]}{s}$ $exists$

$P'(t) = P\ (t) \lim_{s\downarrow 0} \frac{[P\ (s)-I]}{s}$

If $\lim_{s\downarrow 0} \frac{[P(s)-I]}{s}$ $exists.$

3.2 LEMMA:

$P_{FrFr}(t) > 0$ *for all* $t \geq 0$ *and* $i \in s$

Proof:

$P_{FrFr}(0) = 1 > 0$

If $t \downarrow 0$, $P_{FrFr}(t) \rightarrow 1$

For some $t_0 > 0$, $P_{FrFr}(t) > 0$ *for* $0 \leq t \leq t_0$

(3.3)

Choose n such that

$t = \left(\frac{t}{n} n\right), {}^{t}/_{n} \leq t_0$

(3.4)

$P_{FrFk}(t) = \cup_{\alpha\in(0,1]} \alpha(P_{FrFr}(t))_\alpha$

$= \sum_{Fk\in s} \cup_{\alpha\in(0,1]} \alpha(P_{FrFk}(\frac{t}{n}))\ P_{FkFr}(\frac{n-1}{n}\ t)\quad By\ (axiom3)$

$\geq \cup_{\alpha\in(0,1]} \alpha\ P_{FrFr}(\frac{t}{n})\ P_{FrFr}(\frac{n-1}{n}\ t)$

By induction on n≥ $P_{FrFr}(\frac{t}{n})^n_\alpha > 0$

4.DIFFERENTIABILITY OF TRANSITION FUNCTION

4.1 THEOREM:

$\lim_{t\downarrow 0} \frac{1-P_{FrFr}(t)}{t} =$

a) q_{FrFr} *exists but may be* ∞

$= -P'_{FrFr}(0)$

b) $\lim_{t\downarrow 0} \frac{P_{FrFr}(t)}{t} = q_{FrFr} < \infty$ *of* $F_r \neq F_s$

c) For each fuzzy state F_r of a finite state Markov process, q_{FrFr} is finite and is equal to $\sum_{F_r\neq F_s} q_{FrFs}$

d) For all $F_r \in S$ $\sum_{F_r\neq F_s} q_{FrFs} \leq q_{FrFr}$

Proof:

a) We shall first see that for every $F_r \in S$

$$-P'_{FrFr}(0) = \lim_{Fr\to 0} \frac{1 - P_{FrFr}(0)}{t}$$

By Champman Kolmogorov equation

$$P_{FrFr}(t+s) \geq P_{FrFr}(t)P_{FrFr}(s)$$

Taking log on both sides and writing

$$-\log P_{FrFr}(t) = \emptyset(t)$$

we get

$$\emptyset(t+s) \leq \emptyset(t)\emptyset(s) \qquad (3.5)$$

Since $0 \leq P_{FrFr}(t) \leq 1$; $\emptyset(t) \geq 0$

Let

$$q_{FrFs} = \sup_{t>0} \frac{\emptyset(t)}{t}$$

$$\cup_{\alpha\in(0,1]} \alpha\ (q_{FrFs})_\alpha = \sup_{t>0} \cup_{\alpha\in(0,1]} \alpha \frac{\emptyset_\alpha(t)}{t}$$

$$= \bigcup_{\alpha\in(0,1]} \alpha \sup_{t>0} \frac{\emptyset_\alpha(t)}{t}$$

Then $0 \le (q_{FrFs})_\alpha \le \infty$

If $(q_{FrFs})_\alpha < \infty$ there exists $t_0>0$

Such that $\frac{\emptyset_\alpha(t_0)}{t_0} \ge (q_{FrFs})_\alpha - \epsilon$

By (3.5)

$$\emptyset_\alpha(t_0) \le \emptyset_\alpha(nt)\emptyset_\alpha(\delta)$$

$$\le \emptyset_\alpha((n-1)t)+\emptyset_\alpha(t)+\emptyset_\alpha(\delta)$$

$$\le n\emptyset_\alpha(t)+\emptyset_\alpha(\delta)$$

If $(t_0) = nt+\delta; \quad 0 \le \delta \le t, \quad n = 1,2,\dots\dots$

Hence

$$\bigcup_{\alpha\in(0,1]} \alpha\,(q_{FrFs})_\alpha - \epsilon \le \bigcup_{\alpha\in(0,1]} \alpha \frac{\emptyset_\alpha(t_0)}{t_0}$$

$$\le \bigcup_{\alpha\in(0,1]} \alpha \frac{n\emptyset_\alpha(t)+\emptyset_\alpha(\delta)}{t_0}$$

$$\le \bigcup_{\alpha\in(0,1]} \alpha \left[\frac{nt}{t_0}\frac{\emptyset_\alpha(t)}{t} + \frac{\emptyset_\alpha(\delta)}{t_0}\right]$$

But $t \to 0 \Rightarrow \frac{nt}{t_0} \to 1$ and $\emptyset_\alpha(\delta) \to 0$

(By Axiom 5) and $P_{FrFr}(\delta) \to 1$ as $\delta \to 0$

Therefore

$$\bigcup_{\alpha\in(0,1]} \alpha \lim_{t\to 0} \frac{\emptyset_\alpha(t_0)}{t_0} = \bigcup_{\alpha\in(0,1]} \alpha \lim_{t\to 0} \left[\frac{nt}{t_0}\frac{\emptyset_\alpha(t)}{t} + \frac{\emptyset_\alpha(\delta)}{t_0}\right]$$

Hence by the definition of q_{FrFr}

$$\bigcup_{\alpha\in(0,1]} \alpha \overline{\lim_{t\to 0}} \frac{\emptyset_\alpha(t)}{t} \le (q_{FrFr})_\alpha$$

By (3.6)(3.7) and (3.8) we have

$$\bigcup_{\alpha\in(0,1]} \alpha\,(q_{FrFr})_\alpha - \epsilon \le \bigcup_{\alpha\in(0,1]} \alpha \overline{\lim_{t\to 0}} \frac{\emptyset_\alpha(t)}{t}$$

$$\le \bigcup_{\alpha\in(0,1]} \alpha \underline{\lim_{t\to 0}} \frac{\emptyset_\alpha(t)}{t}$$

$$= \bigcup_{\alpha\in(0,1]} \alpha\,(q_{FrFr})_\alpha$$

Since $\epsilon > 0$ is arbitrary we have

$$\bigcup_{\alpha\in(0,1]} \alpha \overline{\lim_{t\to 0}} \frac{\emptyset_\alpha(t)}{t} = \bigcup_{\alpha\in(0,1]} \alpha \underline{\lim_{t\to 0}} \frac{\emptyset_\alpha(t)}{t}$$

$$= \bigcup_{\alpha\in(0,1]} \alpha\,(q_{FrFr})_\alpha$$

If $(q_{FrFr})_\alpha = \infty$ we can replace $(q_{FrFr})_\alpha - \epsilon$ by an arbitrary large constant M and then obtain

(3.6)

$$M \le \bigcup_{\alpha\in(0,1]} \alpha \underline{\lim_{t\to 0}} \frac{\emptyset_\alpha(t)}{t}$$

Thus

$$\infty = \bigcup_{\alpha\in(0,1]} \alpha \underline{\lim_{t\to 0}} \frac{\emptyset_\alpha(t)}{t}$$

(3.7)

In either case we have

$$\bigcup_{\alpha\in(0,1]} \alpha \lim_{t\to 0}\frac{\emptyset_\alpha(t)}{t} \le \bigcup_{\alpha\in(0,1]} \alpha\,(q_{FrFr})_\alpha$$

Now

$$\bigcup_{\alpha\in(0,1]} \alpha \lim_{t\to 0}\frac{1-(P_{FrFr})_\alpha}{t} = \bigcup_{\alpha\in(0,1]} \alpha \lim_{t\to 0}\frac{1-exp(-\emptyset_\alpha(t)}{t}\;\frac{\emptyset_\alpha(t)}{t}$$

$$= \bigcup_{\alpha\in(0,1]} \alpha\,(q_{FrFr})_\alpha$$

$$= q_{FrFr}$$

b) Let $C\in\left(\frac{1}{2},1\right)$ be fixed. Then by lemma (3.2) there exist an S>0 sufficiently small so that

$$(P_{FrFr})_\alpha(S) > C \; and$$

$$(P_{FrFs})_\alpha(S) > C \; for\; \alpha \in (0,1]$$

Let $0 < S \le nt < \delta$. Consider a stationary discrete parameter Markov chain $(X_n)_0^\infty$ with state space S such that

$$P[\,X_{n+1} = F_s\,|X_n = F_r] = P_{FrFs}(t)$$

Then for all n>1 we have

$$P_{FrFs}(nt) = \bigcup_{\alpha\in(0,1]} \alpha\; P_\alpha[\,X_n = F_s\,|X_0 = F_r]$$

$$\ge \bigcup_{\alpha\in(0,1]} \alpha\, P_\alpha[\,X_1 = F_s, X_n = F_r\,|X_0 = F_r] + \bigcup_{\alpha\in(0,1]} \sum_{r=1}^{n-1} P_\alpha[\,X_1 \ne F_s, \dots\dots X_{r-1} \ne F_s, X_r = F_r\,, X_{r+1} = F_s \dots . X_n = F_s\,|X_0 = F_r]$$

$$\ge \bigcup_{\alpha\in(0,1]} \alpha\, C\,(P_{FrFs}(t))_\alpha\;[1+\sum_{r=1}^{n-1} P_\alpha\,(\,X_1 \ne F_s, \dots\dots X_{r-1} \ne F_s, X_r = F_r\;\; |X_0 = F_r)]$$

Since each term on the right corresponds to a possible way of going from Fr to Fs in n-steps and these paths are mutually exclusive but not necessarily exhaustive.

on the other hand

$$\bigcup_{\alpha\in(0,1]} \alpha P_\alpha[\,X_1 \ne F_s, \dots\dots X_{r-1} \ne F_s, X_r = F_r|X_0 = F_r]$$

$$= \bigcup_{\alpha\in(0,1]} \alpha\, P_\alpha[\,X_1 \ne F_s, \dots\dots X_{r-2} \ne F_s, X_r = F_r|X_0 = F_r] - \bigcup_{\alpha\in(0,1]} \alpha\, P_\alpha[\,X_1 \ne F_s, \dots\dots X_{r-2} \ne F_s, X_{r-1} = F_s, X_r = F_r|X_0 = F_r]$$

$$= \bigcup_{\alpha\in(0,1]} \alpha P_\alpha[\,X_1 \ne F_s, \dots\dots X_{r-3} \ne F_s, X_r = F_r|X_0 = F_r] - \bigcup_{\alpha\in(0,1]} \alpha\, P_\alpha[\,X_1 \ne F_s, \dots\dots X_{r-3} \ne F_s, X_{r-2} = F_s, X_r = F_r|X_0 = F_r] - \bigcup_{\alpha\in(0,1]} \alpha P_\alpha[\,X_1 \ne F_s, \dots\dots X_{r-2} \ne F_s X_{r-1} = F_s, X_1 = F_r|X_0 = F_r]$$

And so on. Continuing in a similar way we obtain

$$\bigcup_{\alpha\in(0,1]} \alpha\, P_\alpha[\,X_1 \ne F_s, \dots\dots X_{r-2} \ne F_s, X_r = F_r|X_0 = 1]P_\alpha[X_r = F_r|X_0 = F_r] - \sum_{k=r} P_\alpha\;[X_1 \ne F_s, \dots\dots X_{k-1} \ne F_s, X_k = F_s]\,P_\alpha[X_r = F_r|X_k = F_s]$$

$\ge$ C- (1-C) $\bigcup_{\alpha\in(0,1]} \alpha\; P_\alpha[\,X_1 \ne F_s, \dots\dots X_{k-1} \ne F_s\; X_k = F_s|X_0 = F_r]$

$\ge 2C - 1$

Consequently

$$\bigcup_{\alpha\in(0,1]} \alpha\,(P_{FrFs}(nt))_\alpha \geq 2C - 1n \bigcup_{\alpha\in(0,1]} \alpha\, P_{FrFs}(t)$$

Let h<δ , t<δ and n=[h/t] where [x] is the integer part of x then

$$\bigcup_{\alpha\in(0,1]} \alpha \frac{(P_{FrFs})_\alpha(t)}{t} \leq \bigcup_{\alpha\in(0,1]} \alpha\, P_{FrFs}\left(\frac{\frac{\left(\frac{h}{t}\right)}{t}}{\left(\frac{h}{t}\right) to\ (2c-1)}\right)$$

Letting t→0+ we obtain

$$\lim_{t\to 0+} \bigcup_{\alpha\in(0,1]} \alpha\ Sup \frac{(P_{FrFs})_\alpha}{t} \leq \bigcup_{\alpha\in(0,1]} \alpha(P_{FrFs})_\alpha\,[hC(2C-1)]$$

< ∞ for all t >0

$$\bigcup_{\alpha\in(0,1]} \alpha \lim_{t\to 0+} Sup \frac{(P_{FrFs})_\alpha}{t} \leq \bigcup_{\alpha\in(0,1]} \alpha \lim_{t\to 0+} \frac{inf\,(P_{FrFs})_\alpha}{h(2C-1)C}$$

Which means that

$$\bigcup_{\alpha\in(0,1]} \alpha(q_{FrFs})_\alpha = \bigcup_{\alpha\in(0,1]} \alpha \lim_{t\to 0+} \frac{(P_{FrFs}(t))_\alpha}{t}$$

< ∞ exists

c) We shall prove $q_{FrFr} < \infty$ for a finite state process.

Now

$$1 = P_{FrFs}(h) + \sum_{\substack{Fr=1\\ Fr\neq Fs}}^{N} P_{FrFs}(h)$$

Where N is the number of fuzzy states

Or

$$1 - P_{FrFr}(h) = \sum_{i\neq j}^{N} P_{FrFs}(h)$$

$$1 - \bigcup_{\alpha\in(0,1]} \alpha(P_{FrFr})_\alpha(h) = \sum_{i\neq j}^{N} \alpha(P_{FrFs})_\alpha(h)$$

Dividing by h and letting h→ 0 and by part (a)

$$\bigcup_{\alpha\in(0,1]} \alpha(q_{FrFs})_\alpha = \bigcup_{\alpha\in(0,1]} \alpha \sum_{i\neq j}^{N} (q_{FrFs})_\alpha$$

< ∞

d) Since

$$\bigcup_{\alpha\in(0,1]} \alpha \sum_{Fr\neq Fs}^{\infty} (P_{FrFs})_\alpha(h) = 1 - \bigcup_{\alpha\in(0,1]} \alpha(P_{FrFr})_\alpha(h)$$

$$\bigcup_{\alpha\in(0,1]} \alpha \sum_{Fr\neq Fs}^{N} (P_{FrFs})_\alpha(h) \leq 1 - \bigcup_{\alpha\in(0,1]} \alpha(P_{FrFr})_\alpha(h)$$

for every finite number N of states.

Dividing both sides by h and letting h→ 0 leads to

$$\bigcup_{\alpha\in(0,1]} \alpha \sum_{Fr\neq Fs}^{\infty} (q_{FrFs})_\alpha \leq \bigcup_{\alpha\in(0,1]} \alpha(q_{FrFr})_\alpha$$

Since N is arbitrary and all terms are positive the result follows.

5.KOLMOGROV DIFFERENTIAL DIFFERENCE EQUATION

5.1 DEFINITION:

A state Fr is called instantaneous if $q_{FrFr} = \infty$ and is called stable if $0 \leq q_{FrFr} < \infty$. A state Fr∈S is called Fuzzy absorbing if

$$\bigcup_{\alpha\in(0,1]} \alpha (q_{FrFr})_\alpha = 0 \text{ for all } \alpha$$

5.2 DEFINITION:

A continuous time standard Markov process is called Conservative if

$$\bigcup_{\alpha\in(0,1]} \alpha \sum_{Fr\neq Fs}^{\infty} (q_{FrFs})_\alpha \leq \bigcup_{\alpha\in(0,1]} \alpha (q_{FrFr})_\alpha < \infty$$

for all Fr∈S.

5.1 THEOREM:

In a conservative Markov Process the transition probability function $P_{FrFs}(t)$ is differentiable for all Fr, Fr∈S and t ≥ 0 and they satisfy Kolmogorovs differential equations

a) Backward Equation:

$$P'_{FrFs}(t) = \sum_{Fk\neq Fr} q_{FrFk} P_{FkFs}(t) - q_{FrFr} P_{FrFs}(t) \quad \text{for all } Fr, Fs \in S \text{ and } t \geq 0$$

b) Forward Equation:

$$P_{FrFs}(t) = \sum_{Fk\neq Fr} q_{FkFr} P_{FrFk}(t) - q_{FsFs} P_{FrFs}(t) \quad \text{for all } Fr, Fs \in S \text{ and } t \geq 0$$

Proof:

a) $$P_{FrFs}(t+s) - P_{FrFs}(t) = \bigcup_{\alpha\in(0,1]} \alpha[(P_{FrFs})_\alpha(t+s) - (P_{FrFs})_\alpha(t)]$$

$$= \bigcup_{\alpha\in(0,1]} \alpha [\sum_{Fk\neq s} (P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t) - (P_{FrFs})_\alpha(t)]$$

$$= \bigcup_{\alpha\in(0,1]} \alpha[(P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t) - [1-(P_{FrFr})_\alpha] - (P_{FrFs})_\alpha(t)]$$

Claim

$$\bigcup_{\alpha\in(0,1]} \alpha \lim_{s\downarrow 0} \sum_{Fk\neq Fr} \frac{(P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t)}{s}$$

$$= \bigcup_{\alpha\in(0,1]} \alpha \sum_{Fk\neq Fr}^{\infty} (q_{FrFk})_\alpha (P_{FkFs})_\alpha(t)$$

$$\bigcup_{\alpha\in(0,1]} \alpha \liminf_{s\downarrow 0} \sum_{Fk\neq Fr}^{\infty} \frac{(P_{FrFk})_\alpha(s)}{s} (P_{FkFs})_\alpha(t) \geq \bigcup_{\alpha\in(0,1]} \alpha \sum_{Fk\neq Fr}^{N} (q_{FrFk})_\alpha (q_{FkFs})_\alpha(t) \quad (5.2)$$

For every positive integer N. And hence

$$\bigcup_{\alpha\in(0,1]} \alpha \sum_{Fk\neq Fr}^{\infty} (q_{FrFk})_\alpha (P_{FkFs})_\alpha(t) \leq \bigcup_{\alpha\in(0,1]} \alpha \liminf_{s\downarrow 0} \sum_{Fk\neq Fr}^{\infty} \frac{(P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t)}{s} \quad (5.3)$$

Take N > F_k then

$$\bigcup_{\alpha\in(0,1]} \alpha \sum_{Fk\in s} (q_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t)$$

$$= \bigcup_{\alpha\in(0,1]} \alpha[\sum_{Fk=0}^{N} (P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t) + \sum_{Fk=N+1}^{\infty} (P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t)]$$

$$\leq \bigcup_{\alpha\in(0,1]} \alpha[\sum_{Fk=0}^{N} (P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t) + \sum_{N+1}^{\infty} (P_{FrFk})_\alpha(s)]$$

$$\leq \bigcup_{\alpha\in(0,1]} \alpha[\sum_{Fk=0}^{N} (P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t) + (1 - \sum_{k=1}^{N} (P_{FrFk})_\alpha(s))]$$

$$= \bigcup_{\alpha\in(0,1]} \alpha[\sum_{Fk=0}^{N} (P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t) + 1 - (P_{FrFr})_\alpha(s)]$$

$$- \bigcup_{\alpha\in(0,1]} \alpha \sum_{Fk\neq Fr}^{N} \frac{(P_{FrFk})_\alpha(s)}{s}$$

$$\bigcup_{\alpha\in(0,1]} \alpha \limsup_{s\downarrow 0} \sum_{Fk\neq Fr}^{\infty} \frac{(P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t)}{s}$$

$$\leq \bigcup_{\alpha\in(0,1]} \alpha\left[\sum_{Fk\neq Fr}^{N} (q_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t) + (q_{FrFr})_\alpha - \sum_{Fk\neq Fr}^{N} (q_{FrFk})_\alpha\right]$$

Now letting N→∞ (Markov process is conservative)

$$\leq \bigcup_{\alpha\in(0,1]} \alpha \sum_{Fk\neq Fr}^{\infty} (q_{FrFk})_\alpha(P_{FkFs})_\alpha(t) \qquad (5.4)$$

Combining (5.3) and (5.4) the backward equation as follows from

$$\bigcup_{\alpha\in(0,1]} \alpha \limsup_{s\downarrow 0} \sum_{Fk\neq Fr}^{\infty} \frac{(P_{FrFk})_\alpha(s)}{s}(P_{FkFs})_\alpha(t) \leq \bigcup_{\alpha\in(0,1]} \alpha \sum_{Fk\neq Fr}^{\infty} (q_{FrFk})_\alpha(P_{FkFs})_\alpha(t)$$

$$\leq \bigcup_{\alpha\in(0,1]} \alpha \liminf_{s\downarrow 0} \sum_{Fk\neq Fr}^{\infty} \frac{(P_{FrFk})_\alpha(s)}{s}(P_{FkFs})_\alpha(t)$$

$\bigcup_{\alpha\in(0,1]} \alpha \lim_{s\downarrow 0} \frac{1}{s}\sum_{Fk\neq Fr}^{\infty}(P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t)$ exists and is equal to

$$\bigcup_{\alpha\in(0,1]} \alpha \sum_{Fk\neq Fr}^{\infty} (q_{FrFk})_\alpha(P_{FkFs})_\alpha(t)$$

b) $$P_{FrFs}(s+t) - P_{FrFs}(s) = \bigcup_{\alpha\in(0,1]} \alpha\, [(P_{FrFs})_\alpha(s+t) - (P_{FrFs})_\alpha(s)]$$

$$= \bigcup_{\alpha\in(0,1]} \alpha[\sum_{Fk\in s} (P_{FrFk})_\alpha(s)(P_{FkFs})_\alpha(t) - (P_{FrFs})_\alpha(s)]$$

$$= \bigcup_{\alpha\in(0,1]} \alpha \sum_{Fk\in s} (P_{FrFk})_\alpha(s)[(P_{FkFs})_\alpha(t) - \delta_{ij}]$$

And hence the result (b) can be proved as (a) after dividing by t and letting t →0

6. CONCLUSION:

In this paper we studied that the Fuzzy logic with the theory of Markov system. We introduced the concept of a Markov system with fuzzy states. We attempted the topic differentiability of fuzzy transition probability.

We Presented theorems and definitions where we could do the computations. We defined the probability of fuzzy event and the α-level set of fuzzy transition function. Finally we provide kolmogorovs Difference equation.

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