A GRAPH THEORETICAL APPROACH TO MONOGENIC AND STRONGLY MONOGENIC RIGHT TERNARY N-GROUPS

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Abstract - ***A right ternary near-ring (RTNR) is an algebraic system which is a group under binary addition and a ternary semigroup under ternary multiplication satisfying the right distributive law. A right ternary N-group (RTNG) over a right ternary near-ring N is a generalization of its binary counterpart. In this paper realizing an RTNR as an RTNG $_N N$, the condition for $_N N$ to be monogenic is given. The graph associated with monogenic RTNG is constructed and it is shown that $_N N$ is monogenic iff the graph associated with it is a complete graph. The condition for $_N N$ to be strongly monogenic is also given and the graph associated with it is shown as a complete graph. The values for some of the graph invariants namely the diameter, girth, maximum and minimum degree of both the graphs when N = $\mathbb{Z}_n$ are computed.***

Key Words: *Right ternary near-ring, zero-symmetric RTNR, right ternary N-group, graph, girth.*

1. INTRODUCTION

Graphs are mathematical structures used to model pairwise relations between objects. The powerful combinatorial methods found in graph theory are used to prove fundamental results in other areas of pure mathematics.

In 1988, Beck [1] introduced the concept of a zero divisor graph in the study of commutative rings and later on Livingston [5] described more basic structure of these graphs. Satyanarayana et al [8] studied about prime graphs in rings. In 2013, Das et al [3] has obtained certain values for the diameter, girth, maximum and minimum degree, domination number etc. of the graphs of monogenic semigroups .

In 2011, Daddi and Pawar [2] introduced right ternary near-ring which is a generalization of a near-ring in ternary context. A right ternary N-group (RTNG) [10] over a right ternary near-ring N is a generalization of its binary counterpart. In this paper realizing an RTNR as an RTNG $_N N$, the condition for $_N N$ to be monogenic is given. The graph MG($_N N$) of a monogenic RTNG is constructed. It is proved that $_N N$ is monogenic iff MG($_N N$) is a complete graph. The graph SMG ($_N N$) of a strongly monogenic RTNG is defined and if $_N N$ is strongly monogenic then it is shown that the graph SMG ($_N N$) is a complete graph. As the graph theoretical approach is easier, the condition for $_N N$ to be monogenic and strongly monogenic if N = $\mathbb{Z}_n$ is easily obtained. The values for some of the graph invariants namely the diameter, girth, maximum and minimum degree of both the graphs are also computed.

2. PRELIMINARIES

In this section the basic definitions and results on RTNR, RTNG and graph theory are given.

Definition 2.1 [2] Let N be a non-empty set together with a binary operation + and a ternary operation [ ] : N×N×N → N .Then (N,+,[ ]) is a *right ternary near-ring* (RTNR) if (N, +) is a group , [[xyz]uv] = [x[yzu]v] = [xy[zuv]] =[xyzuv] for every x,y,z,u,v∈N and [(x + y) z w] = [x z w] + [y z w] for every x,y,z,w∈N.

Definition 2.2 [9] If N is an RTNR then N_0= {n ∈N | [n 0 0] = 0} is the *zero-symmetric part* of N .If N = N_0 then N is called *a zero-symmetric RTNR.*

Definition 2.3 [10] Let (N, +, [ ]) be an RTNR and (Γ, +) be a group with additive identity 0_Γ. Then Γ is said to be a *right ternary N-group* if there exists a mapping

$[\ ]_\Gamma : N \times N \times \Gamma \rightarrow \Gamma$ satisfying the conditions

(RTNG-1) $[(n + m)\ x\ \gamma]_\Gamma = [n\ x\ \gamma]_\Gamma + [m\ x\ \gamma]_\Gamma$

(RTNG-2) $[[n\ m\ u]\ x\ \gamma]_\Gamma = [n\ [m\ u\ x]\ \gamma]_\Gamma = [n\ m\ [u\ x\ \gamma]_\Gamma\ ]_\Gamma$ for all $\gamma \in \Gamma$ and n, m, u ∈ N.

Every RTNR is an N-group and is denoted by $_N N$.

A subgroup Δ of $_N\Gamma$ is said to be an *N-subgroup* of $_N\Gamma$ if $[NN\Delta]_\Gamma \subseteq \Delta$

Let Γ be a right ternary N-group. Then for x ∈ N and γ∈ Γ, $_N\Gamma$ is *monogenic* by γ *w.r.to x* if $[Nx\gamma]_\Gamma = \Gamma$ and $_N\Gamma$ is *monogenic by γ* if there exists γ∈ Γ and for every x ∈ N, $[N x \gamma]_\Gamma = \Gamma$.

A right ternary N-group Γ is *strongly monogenic* if Γ is monogenic(*by γ)* and $[Nx\gamma]_\Gamma = \Gamma$ or $\{0_\Gamma\}$ for every x ∈N and γ∈ Γ.

An RTNG is N-simple if its only N-subgroups are $[N00_\Gamma]_\Gamma$ and Γ.

Definition 2.4 [4] A graph is an ordered pair *G* = (V, E) comprising a set V of vertices or nodes together with a set E of edges or lines, which are 2-element subsets of V. The *distance* from u to v in a graph G, denoted dist(u, v), is the shortest length of a *u-v* path in G.

The *diameter* of a graph G is defined by

$$\text{diam}(G) = \max_{u,v\in V(G)} \text{dist}(u,v).$$

The *degree* of a vertex is the number of vertices adjacent to it. A vertex with degree 0 is called an *isolated vertex*. The *maximum degree* of a graph *G*, denoted by Δ(*G*), and the *minimum degree* of a graph, denoted by δ(*G*), are the maximum and minimum degree of its vertices

A *walk* of length *k* is a sequence of vertices $v_0, v_1, \ldots, v_k$, such that for all i > 0, v_i is adjacent to v_{i-1}. A closed walk in any graph that uses every edge exactly once is called an *Euler cycle.* An *Eulerian graph* is a graph containing an Eulerian *cycle.*

A graph is Eulerian if and only if it is a connected graph in which every vertex has even degree.

A *connected* graph is a graph such that for each pair of vertices v_1 and v_2 there exists a walk beginning at v_1 and ending at v_2.

A *totally disconnected* graph is a graph which has only isolated vertices.

A *cycle* of length *k* > 2 is a walk such that each vertex is unique except that $v_0 = v_k$.

The *girth* of a graph is the length of its shortest cycle. If there is no cycle in G, then its girth is ∞.

A graph is *r-regular* if every vertex has degree *r*.

A *complete* graph is a graph such that every pair of vertices is connected by an edge.

Definition 2.5 [6] A *dominating set* of a graph G is a set D of vertices of G such that every vertex of V(G) - D has a neighbour in D. The *domination number* γ(G) is the minimum cardinality of a dominating set in G.

3. A GRAPH OF A MONOGENIC N-GROUP

In this section a zero-symmetric RTNR is regarded as a right ternary N-group and the condition for $_NN$ to be monogenic is given. The graph associated to monogenic RTNG denoted by $MG(_NN)$ is defined and it is proved that $_NN$ is monogenic iff $MG(_NN)$ is complete. The diameter, girth, maximum and minimum degrees for $MG(_NN)$ are calculated when N = $\mathbb{Z}_n$.

Definition 3.1 Let N be an RTNR and realizing N as a right ternary N-group the following definitions are given.

(i) Let x , y ∈N then N is said to be *monogenic* by y *w.r.to* x if [Nxy] = N = [Nyx] where x ≠ y

(ii) N is said to be *monogenic (by y)* if [Nxy] = N = [Nyx] ∀x(≠ y) ∈ N.

(iii) N is said to be *monogenic* if [Nxy] = N = [Nyx] for all non-zero distinct elements x,y ∈ N .

Example 3.2 (i) Let N = {*0, a, b, c, x, y*}be as given in [7, Scheme34, p.411] with + is as defined in Table -1 and [abc] = (a.b).c where. is as defined in Table-2. Then $_NN$ is monogenic.

Table-1

+	*0*	*a*	*b*	*c*	*x*	*y*
0	*0*	*a*	*b*	*c*	*x*	*y*
a	*a*	*0*	*y*	*x*	*c*	*b*
b	*b*	*x*	*0*	*y*	*a*	*c*
c	*c*	*y*	*x*	*0*	*b*	*a*
x	*x*	*b*	*c*	*a*	*y*	*0*
y	*y*	*c*	*a*	*b*	*0*	*x*

Table-2

.	*0*	*a*	*b*	*c*	*x*	*y*
0	*0*	*0*	*0*	*0*	*0*	*0*
a	*0*	*a*	*a*	*a*	*a*	*a*
b	*0*	*b*	*b*	*b*	*b*	*b*
c	*0*	*c*	*c*	*c*	*c*	*c*
x	*0*	*x*	*x*	*x*	*x*	*x*
y	*0*	*y*	*y*	*y*	*y*	*y*

(ii) Let N = S_3 = {*0, a, b, c, x, y*} be as given in [7, Scheme39, p.411]]with + is as defined in Table - 1 and [abc] = (a.b).c where. is defined as in Table -3. Then $_NN$ is monogenic.

Table-3

.	*0*	*a*	*b*	*c*	*x*	*y*
0	*0*	*0*	*0*	*0*	*0*	*0*
a	*a*	*a*	*a*	*a*	*a*	*a*
b	*b*	*b*	*b*	*b*	*b*	*b*
c	*c*	*c*	*c*	*c*	*c*	*c*
x	*x*	*x*	*x*	*x*	*x*	*x*
y	*y*	*y*	*y*	*y*	*y*	*y*

Definition 3.3 Let N be an RTNR .Then define MG ($_N$N) = (V, E) where V = N* = N - {0} and E = {$\overline{xy}$ | [Nxy] = N = [Nyx], x ≠ y}.
Note that to exclude the possibility of having an isolated vertex associated with the zero element the vertex 0 is not included.

Example 3.4 MG ($_N$N) for N in both the examples are complete and is given in Fig.1.

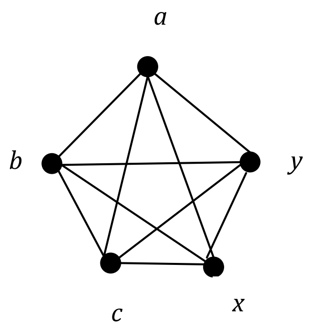

Fig.1

Theorem 3.5 If N is an RTNR then $_N$N is monogenic iff MG($_N$N) is complete.
Proof: Let $_N$N be monogenic. Then for all non-zero distinct elements x,y in N .
[Nxy]=N=[Nyx].This implies that there is an edge between any two distinct elements of N*, showing that the graph MG ($_N$N) is complete.
Conversely, if MG($_N$N) is complete then any two distinct non-zero elements x,y ∈ N are connected and hence by Definition3.3, [Nxy] = N = [Nyx] ∀ x, y ∈ N*. This implies that $_N$N is monogenic.
It can be noted that MG($_N$N) is totally disconnected iff $_N$N is monogenic is not monogenic by an element w.r.to any other element.

Theorem 3.6 In an RTNR N, [Naa] = N and [Nbb] = N for a, b∈N* with a≠ b iff [Nab] = N = [Nba].
Proof: Consider N = [Nbb] = [N[xaa]b] = [[Nxa]ab] ⊆ [Nab]⊆N, which implies that [Nab] = N. Also N = [Naa] = [N[ybb]a] = [[Nyb]ba] ⊆ [Nba]⊆N, which implies that [Nba] = N.
Conversely let[Nab] = N = [Nba].
Then N = [Nba] =[N[tba]a] =[[Ntb]aa] ⊆ [Naa]⊆N. Hence [Naa] = N. Similarly it can be proved that [Nbb] = N.

Theorem3.7 (i) If [Naa] = N ∀ a∈N* then MG($_N$N) is complete. (ii) If N is integral and $_N$N is N- simple then MG($_N$N) is complete.
Proof: (i) Since [Naa] = N ∀ a∈N* by the above lemma [Nab] = N = [Nba] ∀a≠ b.
Hence $_N$N is monogenic and hence by Theorem3.5 MG($_N$N) is complete.

(ii) We note that [Nxy] = {0} only if x = 0 or y = 0 as N is integral. Also [Nxy] = N = [Nyx] as $_N$N is N- simple and integral. Hence MG($_N$N) is complete.

Algorithm to draw the graph MG ($_N$N) 3.8:
Input N*
Output The graph MG($_N$N)
Step1: Find A={x∈N* | [Nxx] = N}
Step2: For x ≠ y in A draw an edge between *x* and *y*
Step 3: Denote the resultant graph as MG ($_N$N).

In the following we consider N = $\mathbb{Z}_n$ ($3 \leq n \leq 10$) and construct the corresponding graph MG ($_N$N).

Construction of MG ($_N$N) where N = $\mathbb{Z}_n$ ($3 \leq n \leq 10$) 3.9:
1. Let N = $\mathbb{Z}_3$ = {0, 1, 2}. Then $\overline{12}$ is the only edge of MG ($_N$N) and the graph is as in Fig.2.
2. Let N = $\mathbb{Z}_4$ = {0, 1, 2, 3}. Then $\overline{13}$ is the only edge of MG ($_N$N) and the graph is as in Fig.3
3. Let N = $\mathbb{Z}_5$ = {0, 1, 2, 3, 4}. Then the edges of MG ($_N$N) are $\overline{12}, \overline{13}, \overline{14}, \overline{23}, \overline{24}, \overline{34}$ and the graph is as in Fig.4.
4. Let N = $\mathbb{Z}_6$ = {0, 1, 2, 3, 4, 5}.Then $\overline{15}$ is the only edge of MG ($_N$N) and the graph is as in Fig.5.
5. Let N = $\mathbb{Z}_7$ = {0, 1, 2, 3, 4, 5, 6}. Then the edges of MG ($_N$N) are $\overline{12}, \overline{13}, \overline{14}, \overline{15}, \overline{16}, \overline{23}, \overline{24}, \overline{25}, \overline{26}, \overline{34}, \overline{35}, \overline{36}, \overline{45}, \overline{46}, \overline{56}$ and the graph is as in Fig.6.
6.Let N = $\mathbb{Z}_8$ = {0,1,2,3,4,5,6,7}. Then the edges of MG($_N$N) are $\overline{13}, \overline{15}, \overline{17}, \overline{35}, \overline{37}, \overline{57}$ and the graph is as in Fig.7.
7.Let N = $\mathbb{Z}_9$ = {0,1,2,3,4,5,6,7,8}. Then the edges of MG($_N$N) *are* $\overline{12}, \overline{14}, \overline{15}, \overline{17}, \overline{18}, \overline{24}, \overline{25}, \overline{27}, \overline{28}, \overline{45}, \overline{47}, \overline{48}, \overline{57}, \overline{58}, \overline{78}$ and the graph is as in Fig.8.
8.Let N = $\mathbb{Z}_{10}$ = {0,1,2,3,4,5,6,7,8,9}. Then the edges of MG($_N$N) are $\overline{13}, \overline{17}, \overline{19}, \overline{37}, \overline{39}, \overline{79}$ and the graph is as in Fig.9.

Fig.2

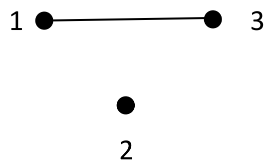

Fig.3

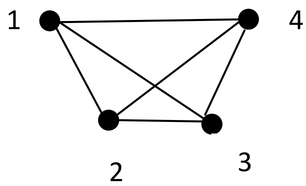

Fig.4

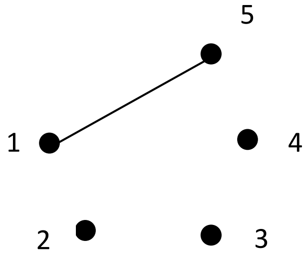

Fig.5

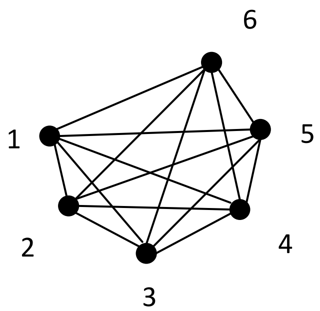

Fig.6

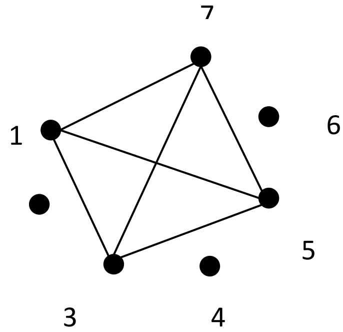

Fig.7

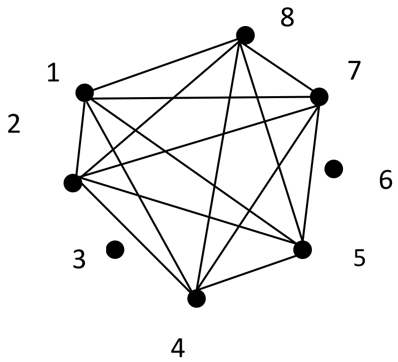

Fig.8

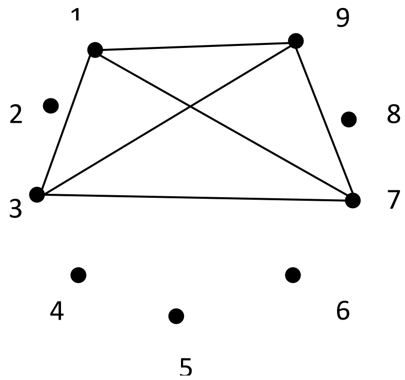

Fig.9

The following properties are observed from the above constructions.

Propertie 3.10

1. MG ($_N$N) is not a connected graph where N = $\mathbb{Z}_n$ ($3 \leq n \leq 10$).

2.There is an edge between i and j iff $(i,n) = 1$ and $(j, n) =1$ where $n = 3,4,...,10$.

3. If n = 5 or 7 then MG($_N$N) is complete.

4. The other observations made from the above examples are given in Table-4

Table-4

n	|E|	Diam (MG($_N$N))	Δ	δ	Girth (MG($_N$N))
3	1	1	1	1	∞
4	1	1	1	0	∞
5	6	1	3	3	3
6	1	1	1	0	∞
7	15	1	5	5	3
8	6	1	3	0	3
9	14	1	5	0	3
10	6	1	3	0	3

For the rest of the section N denotes the RTNR $\mathbb{Z}_n$ = {0,1,2,..., n-1} under the operations addition modulo n and the ternary product [xyz] = (x.y).z and . is multiplication modulo n.

Lemma 3.11 If i, j ∈N* with $(i, n) = 1$ and $(j, n) = 1$ then [Nij] = N

Proof: Let $i, j \in \mathbb{Z}_n$ with $(i, n) = 1$ and $(j, n) = 1$.Then we have the following three cases:

(i) $i.i = 1$, $j.j = 1$ (ii) $i.j = 1 = j.i$ (iii) $i.k = 1$, $j.l = 1$where $k.l \in \mathbb{Z}_n$for some $(i, n) = 1$ and $(j, n) = 1$.

Case (i): If (i) holds then [Nii] ={[nii] |n∈ $\mathbb{Z}_n$} = {(n.i).i | n∈ $\mathbb{Z}_n$} = {n.(i.i)| n∈ $\mathbb{Z}_n$} = N and [Njj] = N and therefore by Theorem 3.6 [Nij] = N.

Case (ii): If (ii) holds then it is obvious that [Nij] = N.

Case(iii): Let $i.k = 1$, $j.l = 1$where $k.l \in \mathbb{Z}_n$with $(k, n) = 1$ and $(l, n) = 1$.Then N = [N11] = [N $i.k$ $j.l$] = [[N k l] i j] ⊆ [N i j] ⊆N.Hence [Ni j] = N.

Now we give an algorithm to draw the graph MG ($_N$N).

Algorithm 3.12

Algorithm to draw the graph MG ($_N$N)

Input N* = $\mathbb{Z}_n$* = { 1, 2,... n-1}, $n \geq 3$

Output The graph MG($_N$N)

Step1: List the units in N and call it as U_n

Step2: Draw an edge between i and j where $i, j \in U_n$ and $i \neq j$.

Step 3: Denote the resultant graph as MG ($_N$N).

Proposition 3.13 If N = $\mathbb{Z}_n$ where $n \geq 3$, then the number of edges of MG($_N$N) is $\frac{\varphi(n)(\varphi(n)-1)}{2}$.

Proof: For $i \neq j$,an edge between i and j of N in MG($_N$N) is drawn if i and j are units in N and since in N there are $\frac{\varphi(n)(\varphi(n)-1)}{2}$ pairs of such i and j's it follows that the number of edges in MG($_N$N) is $\frac{\varphi(n)(\varphi(n)-1)}{2}$.

Proposition 3.14 If N = $\mathbb{Z}_n$ where $n \geq 3$ then the diameter of the graph MG($_N$N) is 1.

Proof: Let N =$\mathbb{Z}_n$, $n \geq 3$.

Case (i): If $n = p$ then as each vertex is adjacent to all the other vertices, diam(MG($_N$N)) = 1.

Case (ii): If $n \neq p$, there will be edges joining the vertices which are relatively prime to n.Hence diam(MG($_N$N)) = 1.

Proposition 3.15 If N = $\mathbb{Z}_n$ where $n \geq 3$ then

$$gr(MG(_NN)) = \begin{cases} 3 & if\ n = 5\ n \geq 7 \\ \infty & if\ n = 3,4,6 \end{cases}$$

Proof: Let N =$\mathbb{Z}_n$={ 0,1, ... , n – 1}, $n \geq 3$.

Case (i): If n = 3 then there is only one edge between 1 and 2 ; If n = 4 then there is only one edge between 1 and 3; If n = 6 then there is only one edge between 1 and 5 and there is no cycle . Hence gr(MG($_N$N) = ∞.

Case (ii): Suppose n = 5. Then by Lemma 3.11, [N12] = N, [N23] = N and [N13]= N. Hence the edges 1—2—3—1 form a triangle. Hence gr(MG($_N$N)) = 3.

Case (iii): If $n \geq 7$ then $\varphi(n) \geq 4$ and hence there exists i, j ∈ $\mathbb{Z}_n$ such that $(i, n) = 1$ and $(j, n) = 1$. Hence by Theorem3.6 [N i j] = N. This implies that the edges 1— i — j — 1 form a triangle. Hence gr(MG($_N$N) = 3.

Proposition 3.16 If N =$\mathbb{Z}_n$ then

(i)Δ(MG($_N$N)) = φ(n) – 1

(ii) $\delta(MG(_NN)) = \begin{cases} p-2 & if\ n = p \\ 0 & otherwise \end{cases}$

Proof: (i) Since the vertices of MG($_N$N) that are connected by edges are the elements of U_n and the number of elements in U_n is the Euler function φ(n) , Δ(MG($_N$N) = φ(n) – 1.

(ii) If $n = p$ then as the vertices of MG($_N$N) that are connected by edges are the elements of U_n and the number of elements in U_n is φ(p) , Δ(MG($_N$N) = p –2.

If $n \neq p$ then the degree of a non-unit is zero and hence δ(MG($_N$N) = 0.

Lemma 3.17 If N =$\mathbb{Z}_p$ where p is a prime number then N is monogenic iff MG($_N$N) is complete.

Proof: Let N = $\mathbb{Z}_p$= { [0], [1],....,[p – 1]}. Then any two distinct elements i,j in N* are relatively prime to p and hence by Lemma 3.11 it follows [Nij] = N. Thus there is an edge between any two distinct non-zero elements of N showing that, MG ($_N$N) is complete or (p – 1) - regular.

Conversely, if MG($_NN$) is complete then any two distinct i,j in N* are connected and hence by Definition3.3, [Nij] = N = [Nji] $\forall\, i, j \in$ N*. This implies that $_NN$ is monogenic.

4. A GRAPH OF A STRONGLY MONOGENIC N-GROUP

In this section the graph of strongly monogenic N-group SMG($_NN$) is constructed by drawing an edge between the vertices x and y such that [Nxy] = [Nyx] = {0} or N where $x \neq y$ and it is shown that if N is strongly monogenic then the graph SMG($_NN$) is a complete graph but the converse is not true. The diameter, girth, maximum and minimum degrees of SMG($_NN$) have been calculated where N = $\mathbb{Z}_n$.

Definition 4.1 Let N be an RTNR and realizing N as a right ternary N-group $_NN$ is *strongly monogenic* if N is monogenic (by an element) and [Nxy] = [Nyx]= {0} or N $\forall$ x , y $\in$ N and $x \neq$ y.

Example 4.2 Let N be as in Example3.2(i) and (ii).Then in both the cases $_NN$ is strongly monogenic.

Definition 4.3 Let Nbe zero-symmetric RTNR. Then define SMG($_NN$) = (V, E) where V = N and E = {$\overline{xy}$ | [Nxy] = [Nyx] = {0} or N, $x \neq y$}.

Example 4.4 SMG ($_NN$) for N in both the examples in Example4.2 are complete and is given in Fig.10.

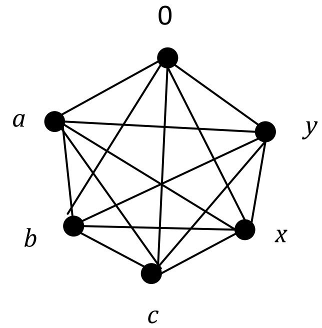

Fig.10

Theorem 4.5 If N is an RTNR and if $_NN$ is strongly monogenic then the graph SMG($_NN$) is complete.
Proof: Let $_NN$ be strongly monogenic. Then 0 is connected to all the other vertices and any two distinct elements x,y $\in$ N are connected. Thus there is an edge between any two elements of N showing that the graph **S**MG($_NN$) is complete.

Remark 4.6 The converse is in general not true. For if N = S_3 = {$0, a, b, c, x, y$} be as given in [7, Scheme1, p.411] with + is as defined in Table - 1 and [abc] = (a.b).c where . is as in Table-5, then SMG($_NN$) is complete but is not strongly monogenic.

Table-5

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	0	a	a	a	a
b	0	0	c	b	c	b
c	0	0	b	c	b	c
x	0	0	y	x	y	x
y	0	0	x	y	x	y

Theorem 4.7 If N is zero-symmetric RTNR and $_NN$ is N-simple then SMG ($_NN$) is complete.
Proof: We note that [Nxy]= {0} if x = 0 or y = 0 as N is zero-symmetric. Also [Nxy] = [Nyx] = N or {0} as $_NN$ is N-simple. Hence SMG($_NN$) is complete.

In the following we consider N= $\mathbb{Z}_n$ ($2 \leq n \leq 10$) and construct the corresponding graph SMG ($_NN$).

Construction of SMG ($_NN$) where N = $\mathbb{Z}_n$ ($2 \leq n \leq 10$)

4.8
1. Let N = $\mathbb{Z}_2$ = {0, 1}. Then there is only one edge $\overline{01}$ in SMG ($_NN$) and the graph is as in Fig.11.
2. Let N = $\mathbb{Z}_3$ = {0, 1, 2}. Then the edges of SMG ($_NN$) are $\overline{01}, \overline{02}, \overline{12}$ and the graph is as in Fig.12.
3. Let N = $\mathbb{Z}_4$ = {0, 1, 2, 3}. Then the edges of SMG ($_NN$) are $\overline{01}, \overline{02}, \overline{03}$, $\overline{13}$ and the graph is as in Fig.13.
4. Let N = $\mathbb{Z}_5$ = {0, 1, 2, 3, 4}. Then the edges of SMG($_NN$) are $\overline{01}, \overline{02}, \overline{03}$, $\overline{04}, \overline{12}, \overline{13}, \overline{14}, \overline{23}, \overline{24}$,$\overline{34}$ and the graph is as in Fig.14.
5.Let N = $\mathbb{Z}_6$ = {0,1,2,3,4,5}.Then the edges of **S**MG($_NN$) are $\overline{01}, \overline{02}, \overline{03}$, $\overline{04}, \overline{05}, \overline{34}, \overline{23}$, $\overline{15}$ and the graph is as in Fig.15
6. Let N = $\mathbb{Z}_7$ = {0,1,2,3,4,5,6}. Then the edges of **S**MG($_NN$) are
$\overline{01}, \overline{02}, \overline{03}$, $\overline{04}, \overline{05}, \overline{06}, \overline{12}, \overline{13}, \overline{14}, \overline{15}, \overline{16}, \overline{23}, \overline{24}, \overline{25}, \overline{26}, \overline{34}, \overline{35}$
$\overline{36}, \overline{45}, \overline{46}, \overline{56}$
and the graph is as in Fig.16.
7.Let N = $\mathbb{Z}_8$ = {0,1,2,3,4,5,6,7}. Then the edges of **S**MG($_NN$) are $\overline{01}, \overline{02}, \overline{03}$, $\overline{04}, \overline{05}, \overline{06}$,$\overline{07}, \overline{13}, \overline{15}, \overline{17}, \overline{35}, \overline{37}, \overline{57}$,$\overline{24}$, $\overline{46}$
and the graph is as in Fig.17.
8.Let N = $\mathbb{Z}_9$ = {0,1,2,3,4,5,6,7,8}. Then the edges of SMG($_NN$)
are $\overline{01}, \overline{02}, \overline{03}$, $\overline{04}, \overline{05}, \overline{06}$,

$\overline{07}, \overline{08}, \overline{36}, \overline{12}, \overline{14}, \overline{15}, \overline{17}, \overline{18}, \overline{24}, \overline{25}, \overline{27}, \overline{28}, \overline{45}, \overline{47}, \overline{48}, \overline{57}, \overline{58}$, $\overline{78}$, and the graph is as in Fig.18.

9.Let N = $\mathbb{Z}_{10}$ = {0,1,2,3,4,5,6,7,8,9}. Then the edges of SMG($_N$N) are $\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{05}, \overline{06}, \overline{07}, \overline{08}, \overline{09}, \overline{13}, \overline{17}, \overline{19}, \overline{37}, \overline{39}, \overline{79}, \overline{25}, \overline{45}, \overline{56}, \overline{58}$ and the graph is as in Fig.19.

Fig.11

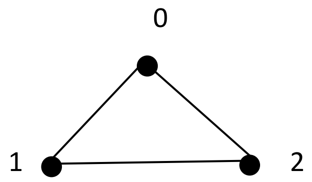

Fig.12

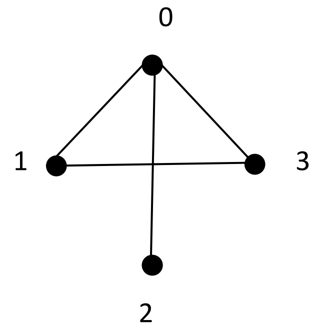

Fig.13

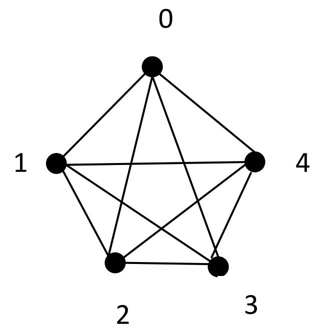

Fig.14

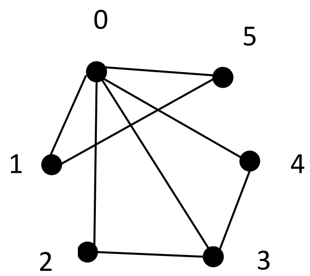

Fig.15

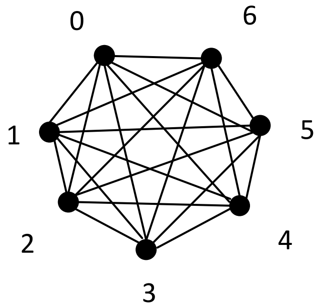

Fig.16

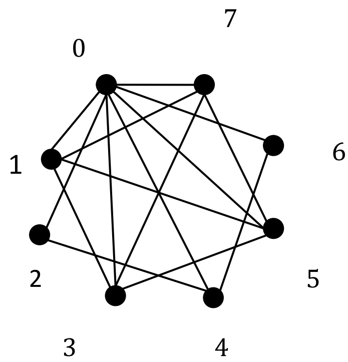

Fig.17

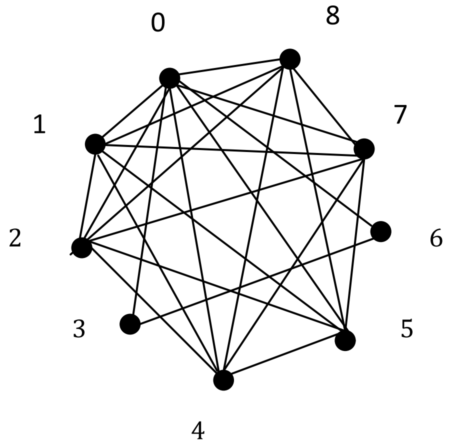

Fig.18

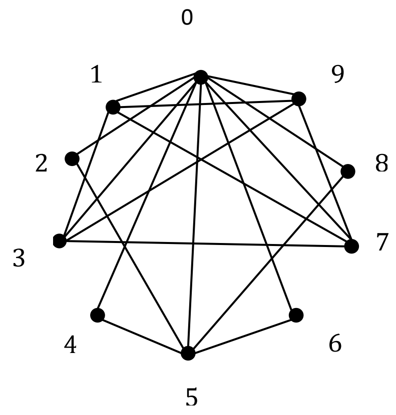

Fig.19

The following properties are observed from the above constructions.

Properties 4.9

1. SMG($_N$N) is a connected graph where N = $\mathbb{Z}_n$(2 $\leq n \leq$ 10).
2. There is an edge between *i* and *j* iff (*i*, *n*) = 1 and (*j*, *n*) =1 where *n* = 2, 3,4,…,10
3. There is an edge between the pairs of zero divisors of N* whose product is
 divisible by *n*.
4. There is an edge between 0 and n ∈N*.
5. SMG ($_N$N) is complete and Eulerian if *n* = 5 or 7.
6. The other observations made from the above examples are given in Table-6.

Table -6

n	|E|	diam (SMG($_N$N)))	Δ	δ	girth(SMG($_N$N))
2	1	1	1	1	∞
3	3	1	2	2	3
4	4	2	3	1	3
5	10	1	4	4	3
6	8	2	5	2	3
7	21	1	6	6	3
8	15	2	7	2	3
9	23	2	8	2	3
10	19	2	9	2	3

Now we give an algorithm to draw the graph **S**MG($_N$N).

Algorithm 4.10
Algorithm to draw the graph SMG($_N$N) where N = $\mathbb{Z}_n$ ={ 0,1, ,...., n – 1}

Input N = $\mathbb{Z}_n$ = {0,1,2, ..., n-1} , $n \geq 2$

Output The graph SMG($_N$N)
Step 1: Find $U_n = \{x \in N^* \mid$ the g.c.d $(x, n) = 1\}$

Step3: For $x,y \in U_n$ with $x \neq y$, draw an edge between x and y .
Step 4: For any $n \in N^*$ draw an edge between 0 and n .
Step 5: Draw an edge between the pairs of zero divisors of $\mathbb{Z}_n$ whose product is divisible by n.
Step 6: Denote the resultant graph as SMG($_N$N)

In what follows N denotes the RTNR $\mathbb{Z}_n$ = {0,1,2,..., n-1} under the operations addition modulo n and the ternary product [xyz] = (x.y).z and . is multiplication modulo n.

Proposition 4.11 If N = $\mathbb{Z}_n$, $n \geq 2$ then the number of edges in SMG($_N$N) is

$$|E| = n - 1 + \frac{\varphi(n)(\varphi(n)-1)}{2} + d.$$

where d is the number of pairs of zero divisors whose product is divisible by n.

Proof: The following are the possibilities to draw an edge between i and j of N in SMG($_N$N) :
(i) i and j are units in N*
(ii) $i = 0$ and $j \neq 0$
(iii) i and j are zero divisors such that $i.j \equiv 0 \pmod{n}$.

Case (i): If i and j are units in N and since in N there are $\frac{\varphi(n)(\varphi(n)-1)}{2}$ pairs of such i and j's , it follows that there are $\frac{\varphi(n)(\varphi(n)-1)}{2}$ edges.

Case (ii): If $i = 0$ and j is any other number then there are (n -1) such edges.

Case(iii): Let i and j be zero divisors whose product is divisible by n and let the number of such pairs be denoted by d.Then in this case there are d edges.
Hence the number of edges in SMG($_N$N) is $|E| = n - 1 + \frac{\varphi(n)(\varphi(n)-1)}{2} + d$.

Remark 4.12. Using the table given in [6], the number of edges of SMG($_N$N) for specific values of n are given below:

1. If $n = p^2$ (p is a prime number greater than or equal to 5) then the number of edges of SMG($_N$N) is
$|E| = p^2 - 1 + \frac{\varphi(p^2)(\varphi(p^2)-1)}{2} + (p-1)C_2$.

2. If $n = 2^2p$, p is an odd prime then the number of edges of SMG($_N$N) is
$|E| = 2^2p - 1 + \frac{\varphi(2^2p)(\varphi(2^2p)-1)}{2} + 4p - 4$.

3. If $n = pq$ where p and q are distinct prime numbers then the number of edges of SMG($_N$N) is
$|E| = pq - 1 + \frac{\varphi(pq)(\varphi(pq)-1)}{2} + (p-1)(q-1)$.

Proposition 4.13 If N = $\mathbb{Z}_n$, $n \geq 2$ is strongly monogenic then SMG($_N$N) is connected.
Proof: Let N be strongly monogenic. Let $i,j \in$N. Then there are 4 cases namely :
(i) $i = 0, j \neq 0$, (ii) $(i, n) = 1$; $(j, n) = 1$,
(iii) $(i, n) = 1$; $(j, n) \neq 1$, (iv) $(i, n) \neq 1$; $(j, n) \neq 1$.

In Case (i) and Case (ii) there is an edge between i and j.
In Case (iii) and Case (iv) i and j will be connected through 0.
Hence there is either one edge or two edges to connect any two elements of N. Thus SMG($_N$N) is connected.

Proposition 4.14 If N = $\mathbb{Z}_n$ ={ 0,1, ,...., n – 1} where $n \geq 2$ then diam(SMG($_N$N)) = $\begin{cases} 1 & if\ n = p \\ 2 & otherwise \end{cases}$

Proof : Let N = $\mathbb{Z}_n$ ={ 0,1, ,...., n – 1} where $n \geq 2$
Case (i): If $n = p$ then as each vertex is adjacent to all the other vertices , diam(SMG($_N$N)) = 1.

Case (ii): If $n \neq p$. Then as discussed in the above proposition there is either one edge or two edges to connect any two elements of N . Hence diam (SMG (N)) = 2.

Proposition 4.15 If N = $\mathbb{Z}_n$ ={ 0,1, ,...., n – 1} where n ≥ 2 then

$$\text{gr(SMG}(_N\text{N})) = \begin{cases} 3\ if\ n \neq 2 \\ \infty\ if\ n = 2 \end{cases}$$

Proof: Let N = $\mathbb{Z}_n$ ={ 0,1, ,...., n – 1} , n ≥ 2.

Case (i): If $n = 2$ then there is only one edge between 0 and 1 and there is no cycle. Hence gr(MG(N) = ∞.

Case (ii): If $n \geq 3$ then $\varphi(n) \geq 2$ and therefore there exists $i, j \in \mathbb{Z}_n$ with $i \neq j$ such that $(i, n) = 1$ and $(j, n) = 1$ .Hence by Lemma3.11 [Nij] = N. This implies that the edges $0 - i - j - 0$ form a triangle. Hence gr(SMG(N) = 3.

Proposition 4.16 If N=$\mathbb{Z}_n$, $n \geq 2$ then γ(SMG(N))= 1.

Proof: Obviously {0} is the dominating set with least number of elements and hence γ (SMG($_N$N)) = 1.

Lemma 4.17 If N = $\mathbb{Z}_p$, p is a prime number then $_N$N is strongly monogenic iff the graph SMG($_N$N) is complete .

Proof: Let N = $\mathbb{Z}_p$. Then 0 is connected to all the other vertices. Also (i, p) =1 and (j, p) = 1 where $i \neq j$. Therefore by Lemma3.11 and Definition 4.1 it follows that i and j are connected. Thus there is an edge between any two elements of N showing that the graph of N is complete. Moreover if i is any arbitrary vertex then as i is connected to all the other (p -1) vertices , SMG(N) is (p – 1) - regular.

Conversely, if SMG($_N$N) is complete then any two distinct non-zero elements i and j in N are connected and hence by Definition 4.1, [Nij] = N or {0} ∀ i, j ($\neq 0$) ∈ N but since [Nij]≠ {0}, [Nij] = N .This implies that N is monogenic. Moreover 0 is connected to all the other vertices and hence [N0j] = {0}.Thus [Ni j] = {0} or N ∀ i, j ∈ N showing that N is strongly monogenic.

Proposition 4.18 If N = $\mathbb{Z}_p$, p is an odd prime number then SMG($_N$N) is Eulerian.

Proof: By the above lemma SMG($_N$N) is complete and each vertex is of degree (p – 1) , an even number and hence SMG($_N$N) is Eulerian.

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Proposition 4.19 If N = $\mathbb{Z}_n$ n ≥ 2 then

1. Δ(SMG($_N$N) = n-1 .

2. δ(SMG($_N$N))= $\begin{cases} p-1 & if\ n = p\ or\ p^2 \\ p & if\ n = p^a q^b r^c \ldots \end{cases}$

Proof: 1. Since 0 is connected to all the other vertices of N, Δ(SMG($_N$N)) = n - 1.

2. Case(i): If n = p then by Lemma 4.17, SMG($_N$N) is complete and hence δ(SMG($_N$N)) = Δ(SMG($_N$N)) = p - 1 .
If $n = p^2$ then δ(SMG($_N$N)) = p – 1, as the prime number p will be incident only with 0, $2(n/p)$, $3(n/p)$,..., $(p - 1) (n/p)$.
Case(ii): If $n = p^a q^b r^c \ldots$ then δ(SMG($_N$N)) = p, as the prime number p will be incident only with 0, n/p, $2(n/p)$, $3(n/p)$,..., $(p - 1) (n/p)$.
The properties of MG($_N$N) and SMG($_N$N) are summarized in the following table :

Table-7

Graph invariants	**MG($_N$N)**	**SMG($_N$N)**
Number of edges	$\frac{\varphi(n)(\varphi(n)-1)}{2}$.	$\lvert E\rvert = n-1+\frac{\varphi(n)(\varphi(n)-1)}{2}$ + the no. of pairs of zero divisors whose product is divisible by n .
Diameter	1	$\begin{cases} 1 & if\ n = p \\ 2 & otherwise \end{cases}$
Δ	φ(n) – 1	n - 1
δ	$\begin{cases} p-2 & if\ n = p \\ 0 & otherwise \end{cases}$	$\begin{cases} p-1 & if\ n = p\ or\ p^2 \\ p & if\ n = p^a q^b r^c \ldots \end{cases}$
girth	$\begin{cases} 3 & if\ n = 5\ n \geq 7 \\ \infty & if\ n = 3,4,6 \end{cases}$	$\begin{cases} \infty\ if & n = 2 \\ 3 & otherwise \end{cases}$
γ	nil	1

5. CONCLUSIONS

The graph associated with an RTNG $_N N$ enabled us to find whether $_N N$ is monogenic or not. Even though the conventional method can be used to check the monogenicity the graph theoretical approach is easier to visualise and the computation is simple. Some of the algebraic properties of $N = \mathbb{Z}_n$ are obtained by studying the graphs $MG(_N N)$ and $SMG(_N N)$. Some more graph properties of MG ($_N N$) and SMG ($_N N$) may be explored to know further about the algebraic properties of $_N N$.

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