International Research Journal of Engineering and Technology (IRJET) e-ISSN: 2395 -0056
Volume: 02 Issue: 06 | Sep-2015 www.irjet.net p-ISSN: 2395-0072

EXACT SOLUTIONS FOR TIME FRACTIONAL COUPLED WHITHAM-BROER-KAUP EQUATIONS VIA EXP-FUNCTION METHOD

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Abstract—In this paper, we used the Exp-function method for solving the time fractional coupled Whitham Broer-Kaup (WBK) equations in the sense of modified Riemann-Liouville derivative. Whit the aid of the mathematical software Maple, some exact solutions for this system are successfully

Keywords—Exp-function method, exact solutions, modified Riemann-Liouville derivative, time fractional coupled Whitham Broer-Kaup (WBK) equations.

INTRODUCTION

Importance of fractional differential equations in studies some natural phenomena, has spurred many researchers for the study and discusses some of the well-known classical differential equations, by replacing some its derivatives or all by fractional derivatives. In this paper we have considered the time fractional coupled Whitham-Broer Kaup equations:[2,3]

$$\begin{cases} D_t^{\alpha}u + uD_x^{\alpha}u + D_t^{\alpha}v + bD_x^{2\alpha}u = 0, \\ D_t^{\alpha}v + D_x^{\alpha}(uv) + aD_x^{3\alpha}u - bD_x^{2\alpha}v = 0, \end{cases} \quad 0 < \alpha < 1 \quad (1)$$

Where their derivatives are the modified Riemann-Liouville derivatives of orderα. These equations is a transformed generalization of the WBK equations [1]. The WBK equations can be used to describe the dispersive long wave in shallow water, when$\alpha = 1$,$b \neq 0$, $a = 0$, sys. (1), is the classical long wave equations that describe the shallow water wave with diffusion. When$\alpha = 1$,$b = 0$,$a = 1$, sys. (1), reduces to the variant Boussinesq equations. In [2] the author solved Sys (1) by projective Riccati equation method, and established some exact solutions for them, and we in this work will apply the described method above. This paper is arranged as follows: In Section 2, we present concepts that we need them to convert the proposed (NFPDE) into a (ODE). In Section 3, we give the description for main steps of the Exp-function method. In Section 4, we apply this method to finding exact solutions for the time fractional coupled Whitham-Broer Kaup equations.

PRELIMINARIES

In this section we list the definition and some important properties of Jumarie's modified Riemann-Liouville derivatives of order α as follows:

Definition 2.1 Let $f(t)$ be a continuous real (but not necessarily differentiable**)** function and let $h > 0$ denote a constant discretization. Then the Jumarie's modified Riemann-Liouville derivative is defined as [4, 6]:

$$D_t^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_0^t (t-u)^{-\alpha-1}(f(u)-f(0))du, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-u)^{-\alpha}(f(u)-f(0))du, & 0 < \alpha < 1, \\ (f^{n}(t))^{\alpha-n}, & n \leq \alpha < n+1, \end{cases} \quad (2)$$

Where

$$D_t^{\alpha}f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{k=0}^{\infty}(k)^{-1}f[x+(\alpha-k)h], \quad (3)$$

In addition, some properties for the proposed modified Riemann-Liouville derivatives are given as follows

$$D_t^{\alpha}t^{r} = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \quad (4)$$

$$D_t^{\alpha}(f(t)g(t)) = g(t)D_t^{\alpha}f(t) + f(t)D_t^{\alpha}g(t), \quad (5)$$

$$D_t^{\alpha}f[g(t)] = f_g[g(t)]D_t^{\alpha}g(t) = D_g^{\alpha}f[g(t)](g'(t))^{\alpha}, \quad (6)$$

This is the direct consequence of the following equation:

$$D_t^{\alpha}f(t) = \Gamma(1+\alpha)Df(t). \quad (7)$$

OUTLINE OF THE EXP-FUNCTION METHOD

In this section we gave a brief description for the main steps of the Exp-function method. For that, consider a nonlinear fractional equation of two independent variables x, t and a dependent variable u of the form [7, 11].

$$P(u, D_t^{\alpha}u, D_x^{2}u, D_x^{3}u, \dots) = 0 \quad (8)$$

Step 1: Firstly, we consider the following transformations;

$$u(x,t) = u(\xi), \quad \xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \quad (9)$$

Where k, c are constants to be determined.

Using Eq. (9) with help Eqs. (4-6) reduces Eq. (8) into an ODE:

$$Q(u, u', u'', u''', \dots) = 0. \quad (10)$$

Step 2: We assume that the solution of the Eq. (10) can be expressed in the form

$$u(\xi) = \frac{\sum_{n=-d}^{c} a_n e^{n\xi}}{\sum_{m=-q}^{p} b_m e^{m\xi}}, \quad (11)$$

Where c, d, p, and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. We can rewrite Eq. (11) in the following form

$$u(\xi) = \frac{a_c e^{c\xi} + \dots + a_{-d} e^{-d\xi}}{b_p e^{p\xi} + \dots b_{-q} e^{-q\xi}}. \quad (12)$$

Step 3: Balancing the linear term of highest order of equation Eq. (10) with the highest order nonlinear term, which leads to $p = c$. Similarly, balancing the linear term of lowest order of Eq. (10) with lowest order nonlinear term, which leads to $d = q$.

Step 4: By substituting (12) into (10), collecting terms of the same term of exp $(i\xi)$, and equating the coefficient of each power of exp to zero, we can get a set of algebraic equations for determining unknown constants.

SOLUTION PROCEDURE

In this section, we apply the Exp-function method for solving the nonlinear time fractional Whitham-Broer-Kaup (WBK) equations.

Example 4.1 Consider the nonlinear time fractional Whitham-Broer-Kaup (WBK) equations:

$$\begin{cases} D_t^{\alpha}u + uD_x^{\alpha}u + D_t^{\alpha}v + bD_x^{2\alpha}u = 0, \\ D_t^{\alpha}v + D_x^{\alpha}(uv) + aD_x^{3\alpha}u - bD_x^{2\alpha}v = 0, \end{cases} \quad 0 < \alpha < 1 \quad (13)$$

Using Eq. (9) along with Eqs. (4-6) So Sys. (13), turns to the following system of (ODEs):

$$\begin{cases} cu' - kuu' - kv' - k^2bu'' = 0 \\ cv' - k(uv)' - ak^3u''' + bk^2v'' = 0 \end{cases} \quad (14)$$

Integrating the first equation of Sys. (14) and neglecting the constant of integration we get

$$v = \frac{c}{k}u - \frac{u^2}{2} - bku'. \quad (15)$$

Substituting Eq. (15) into second equation of Sys. (14), we get

$$(a+b^2)k^4u''' - k^2u^2u' + 3ckuu' - c^2u = 0. \quad (16)$$

Integrating Eq. (16) with zero constant of integration, we find

$$(a+b^2)k^4u'' - \frac{k^2}{2}u^3 + \frac{3}{2}cku^2 - c^2u = 0. \quad (17)$$

Assume that the solution of Eq. (17) can be expressed in the form

$$u(\xi) = \frac{\sum_{n=-d}^{c} a_n e^{n\xi}}{\sum_{m=-q}^{p} b_m e^{m\xi}} = \frac{a_c e^{c\xi} + \dots + a_{-d} e^{-d\xi}}{b_p e^{p\xi} + \dots b_{-q} e^{-q\xi}}. \quad (18)$$

In order to determine the values of c and p, we balance the linear term of highest order in Eq. (17) with the highest nonlinear term. By simple calculation, we have

$$u'' = \frac{c_1 e^{(c+3p)\xi} + \dots}{c_2 e^{4p\xi} + \dots}, \quad (19)$$

And

$$u^3 = \frac{c_3 e^{3cp\xi} + \dots}{c_4 e^{3p\xi} + \dots} = \frac{c_1 e^{(c+3p)\xi} + \dots}{c_2 e^{4p\xi} + \dots}, \quad (20)$$

Where c_1, c_2, c_3 and c_4 are determined coefficients only for simplicity. Balancing highest order of exp-function in Eq. (19) and Eq. (20) we have

$$c + 3p = 3c + p,$$

This leads to the result $p = c$. Similarly to determine the values of d and q we balancing the linear term of lowest order in Eq. (17) with the lowest order nonlinear term. By simple calculation, we have

$$u'' = \frac{\dots + d_1 e^{-(d+3q)\xi}}{\dots + d_2 e^{-4p\xi}}, \quad (21)$$

And

$$u^3 = \frac{\dots + d_3 e^{-3d\xi} + \dots}{\dots + d_4 e^{-3q\xi} + \dots} = \frac{\dots + d_1 e^{-(d+3q)\xi}}{\dots + d_2 e^{-4p\xi}}, \quad (22)$$

Where d_1, d_2, d_3 and d_4 are determined coefficients only for simplicity. From Eq. (21) and Eq. (22) we have

$$-d - 3q = -3d - q,$$

And this gives $q = d$. The values of c, d, can be freely chosen. So for simplicity we investigate three cases:

Case 1: If $p = c = 1, q = d = 1$, Eq. (18) becomes

$$u(\xi) = \frac{a_1 e^{\xi} + a_0 + a_{-1} e^{-\xi}}{b_1 e^{\xi} + b_0 + b_{-1} e^{-\xi}}. \tag{23}$$

In the case $b_1 \neq 0$, Eq. (23) can be simplified as

$$u(\xi) = \frac{a_1 e^{\xi} + a_0 + a_{-1} e^{-\xi}}{e^{\xi} + b_0 + b_{-1} e^{-\xi}}. \tag{24}$$

Substituting Eq. (24) into Eq. (17) and taking the coefficients of e^{ξ} in each term zero yields to a set of algebraic equations for a_{-1}, a_0, a_{-1}, b_{-1}, b_0, c and k as follows:

$$e^{-3\xi}: k^2 a_{-1}^3 - 3cka_{-1}^2 b_{-1} + 2c^2 a_{-1} b_{-1}^2 = 0,$$

$$\begin{aligned} e^{-2\xi}: &2c^2 a_0 b_{-1}^2 - 2k^4 a a_0 b_{-1}^2 - 2k^4 b^2 a_0 b_{-1}^2 + 3k^2 a_0 a_{-1}^2 \\ &3cka_{-1}^2 b_0 - 2k^4 b^2 a_{-1} b_0 b_{-1} + 6cka_0 a_{-1} b_{-1} \\ &-4c^2 a_{-1} b_0 b_{-1} - 2k^4 a a_{-1} b_0 b_{-1} = 0, \end{aligned}$$

$$\begin{aligned} e^{-\xi}: &-2k^4 a a_{-1} b_0^2 + 8k^4 a a_{-1} b_{-1} + 2k^4 b^2 a_0 b_{-1} b_0 \\ &-6cka_1 a_{-1} b_{-1} - 6cka_0 a_{-1} b_0 + 4c^2 a_0 b_{-1} b_0 \\ &-8k^4 a a_1 b_{-1}^2 + 3k^2 a_1 a_{-1}^2 - 3cka_{-1}^2 + 2c^2 a_1 b_{-1}^2 \\ &+4c^2 a_{-1} b_{-1} - 2k^4 b^2 a_{-1} b_0^2 + 3k^2 a_0^2 a_{-1} + \\ &2k^4 a a_0 b_{-1} b_0 - 8k^4 k^4 b^2 a_1 b_0^2 + 8k^4 b^2 a_{-1} b_{-} \\ &-3cka_0^2 b_{-1} = 0, \end{aligned}$$

$$\begin{aligned} e^{0\xi}: &2c^2 a_0 b_0^2 - 6cka_1 a_0 b_{-1} - 6k^2 a a_1 b_0 b_{-1} \\ &-6cka_1 a_{-1} b_0 - 6k^4 b^2 a_{-1} b_0 - 6k^2 b^2 a_1 b_0 b_{-1} \\ &+12k^4 b^2 a_0 b_{-1} + 6k^4 a_1 a_0 a_{-1} - 6cka_0 a_{-1} \\ &+k^2 a_0^2 - 3cka_0^2 b_0 + 4c^2 a_{-1} b_0 + 4c^2 a_0 b_{-1} \\ &-6k^4 a a_{-1} b_0 + 12k^4 a a_0 b_{-1} + 4c^2 a_1 b_0 b_{-1} = 0, \end{aligned}$$

$$\begin{aligned} e^{\xi}: &-2k^4 a a_1 b_0^2 + 2k^4 a a_0 b_0 - 2k^4 b^2 a_1 b_0^2 + 8k^4 b^2 a_1 b_{-1} \\ &+2c^2 a_{-1} - 8k^4 b^2 a_{-1} + 3k^2 a_1^2 a_{-1} + 2k^4 b^2 a_0 b_0 \\ &+3k^2 a_1 a_0^2 + 8k^4 a a_1 b_{-1} - 3cka_0^2 + 2c^2 a_1 b_0^2 \\ &+4c^2 a_1 b_{-1} - 6cka_1 a_0 b_0 + 4c^2 a_0 b_0 - 8k^4 a a_{-1} \\ &-3cka_1^2 b_{-1} - 6cka_1 a_{-1} = 0, \end{aligned}$$

$$\begin{aligned} e^{2\xi}: &6cka_1 a_0 - 2c^2 a_0 + 2k^4 b^2 a_0 - 4c^2 a_1 b_0 - 3k^2 a_1^2 a_0 \\ &2k^4 a a_0 - 2k^4 b^2 a_1 b_0 - 2k^4 a a_1 b_0 + 3cka_1^2 b_0 = 0, \end{aligned}$$

$$e^{3\xi}: 3\,cka_1^2 - k^2 a_1^3 - 2c^2 a_1 = 0.$$

Solving these equations with the aid of Maple we get the five sets of solutions as follows:

$$\mathbf{1.}\ a_{-1} = 0, a_0 = 0, a_1 = \pm 4k\sqrt{a + b^2},$$

$$b_{-1} = b_{-1}, b_0 = 0, c = \pm 2k^2\sqrt{a + b^2}, k = k. \tag{25}$$

$$\mathbf{2.}\ a_{-1} = 0, a_0 = \pm 2kb_0\sqrt{a + b^2}, a_1 = 0,$$

$$b_{-1} = b_{-1}, b_0 = b_0, c = \pm k^2\sqrt{a + b^2}, k = k. \tag{26}$$

$$\mathbf{3.}\ a_{-1} = 0, a_0 = a_0, a_1 = \pm 2k\sqrt{a + b^2},$$

$$b_{-1} = \frac{\pm 4ka_0 b_0\sqrt{a+b^2} - a_0^2}{4k^2(a+b^2)}, b_0 = b_0,$$

$$c = \pm k^2\sqrt{a + b^2}, k = k. \tag{27}$$

$$\mathbf{4.}\ a_{-1} = \pm 4kb_{-1}\sqrt{a + b^2}, a_0 = 0, a_1 = 0,$$

$$b_{-1} = b_{-1}, b_0 = 0, c = \pm 2k^2\sqrt{a + b^2}, k = k. \tag{28}$$

$$\mathbf{5.}\ a_{-1} = \frac{\pm 2ka_0 b_0\sqrt{a + b^2} - a_0^2}{2k\sqrt{a + b^2}}, a_0 = a_0, a_1 = 0,$$

$$b_{-1} = \frac{\pm 2ka_0 b_0\sqrt{a+b^2} - a_0^2}{4k^2(a+b^2)}, b_0 = b_0,$$

$$c = \pm k^2\sqrt{a + b^2}, k = k. \tag{29}$$

Substituting Eqs. (25 - 29) into Eq. (24) we obtain respectively the following solutions:

$$u_1(\xi) = \frac{4k\sqrt{a+b^2}e^{\xi}}{e^{\xi}+b_{-1}e^{-\xi}},$$

$$v_1(\xi) = \frac{8k^2 b_{-1}\left(a+b^2-b\sqrt{a+b^2}\right)}{\left(e^{\xi}+b_{-1}e^{-\xi}\right)^2}. \tag{30}$$

$$u_2(\xi) = \frac{2kb_0\sqrt{a+b^2}}{e^{\xi}+b_0},$$

$$v_2(\xi) = \frac{2k^2 b_0\left(a+b^2+b\sqrt{a+b^2}\right)e^{\xi}}{\left(e^{\xi}+b_0\right)^2}. \tag{31}$$

$$u_3(\xi) = \frac{1}{\varphi}\left(2k\sqrt{a + b^2}e^{\xi} + a_0\right),$$

$$v_3(\xi) = \frac{1}{\varphi}\left(2k^2(a + b^2)e^{\xi} + a_0 k\sqrt{a + b^2}\right)$$

$$-\frac{1}{2\varphi^2}\left(2k\sqrt{a+b^2}e^{\xi}+a_0\right)^2-\frac{2bk^2}{\varphi}\left(\sqrt{a+b^2}e^{\xi}\right)+\frac{bk}{\varphi^2}$$

$$\left(2k\sqrt{a+b^2}e^{\xi}+a_0\right)\left(e^{\xi}-\frac{(2a_0b_0k\sqrt{a+b^2}-a_0^2)e^{-\xi}}{4k^2(a+b^2)}\right) \qquad (32)$$

$$u_4(\xi)=\frac{4b_{-1}k\sqrt{a+b^2}e^{-\xi}}{e^{\xi}-b_{-1}e^{-\xi}},$$

$$v_4(\xi)=\frac{8k^2b_{-1}(a+b^2+b\sqrt{a+b^2})}{(e^{\xi}-b_{-1}e^{-\xi})^2}. \qquad (33)$$

$$u_5(\xi)\frac{1}{\varphi}\left(a_0+\frac{(2a_0b_0k\sqrt{a+b^2}-a_0^2)e^{-\xi}}{2k(\sqrt{a+b^2})}\right),$$

$$v_5(\xi)=\frac{1}{\varphi}\left(a_0k\sqrt{a+b^2}+\frac{1}{2}(2a_0b_0k\sqrt{a+b^2}-a_0^2)e^{-\xi}\right)$$

$$-\left(\frac{a_0}{\varphi^2}+\frac{(2a_0b_0k\sqrt{a+b^2}-a_0^2)e^{-\xi t^{\alpha}}}{2\varphi^2k(\sqrt{a+b^2})}\right)^2$$

$$+\frac{b(2a_0b_0k\sqrt{a+b^2}-a_0^2)}{2\varphi\sqrt{a+b^2}}e^{-\xi}$$

$$+\frac{b}{8\varphi^2k^2(a+b^2)\sqrt{a+b^2}}\Big[2a_0k\sqrt{a+b^2}$$

$$+(2a_0b_0k\sqrt{a+b^2}-a_0^2)e^{-\xi}$$

$$\times\left(4k^2(a+b^2)e^{\xi}-(2a_0b_0k\sqrt{a+b^2}-a_0^2)e^{-\xi}\right)\Big]. \qquad (34)$$

Where

$$\varphi=e^{\xi}+b_0+\frac{(2a_0b_0k\sqrt{a+b^2}-a_0^2)e^{-\xi}}{4k^2(a+b^2)}, \qquad u(\xi)=u(x,t),$$

$$\xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{ct^{\alpha}}{\Gamma(1+\alpha)},$$

Case 2: For the case with p = c = 2, q = d =1, Eq. (18) becomes

$$u(\xi)=\frac{a_2e^{2\xi}+a_1e^{\xi}+a_0+\cdots+a_{-1}e^{-\xi}}{e^{2\xi}+b_1e^{\xi}+b_0\ldots b_{-1}e^{-\xi}} \qquad (35)$$

Substituting Eq. (35) into Eq. (17) and taking the coefficients of e^{ξ} in each term zero yields to a set of algebraic equations for a_{-1}, a_0, a_1, a_2, b_{-1}, b_0, b_1, c and k as follows:

$$e^{-3\xi}: k^4a_{-1}^3+3cka_{-1}^2b_{-1}-2c^2a_{-1}b_{-1}^2=0,$$

$$\begin{aligned}e^{-2\xi}:\ &-2k^4aa_{-1}b_0b_{-1}+6cka_0a_{-1}b_{-1}-2c^2a_0b_{-1}^2\\&-2k^4b^2a_{-1}b_0b_{-1}+2k^4aa_0b_{-1}^2+2k^4b^2a_0b_{-1}^2\\&-3k^2a_0b_{-1}^2+3cka_{-1}^2b_0-4c^2a_{-1}b_0b_{-1}=0,\end{aligned}$$

$$\begin{aligned}e^{-\xi}:\ &-2k^4b^2a_0b_{-1}b_0+6cka_1a_{-1}b_{-1}+3cka_{-1}^2b_1\\&-8k^4b^2a_{-1}b_1b_{-1}-8k^4aa_{-1}b_1b_{-1}+8k^4b^2a_1b_{-1}^2\\&-2k^4aa_0b_{-1}b_0-4c^2a_{-1}b_1b_{-1}+8k^4aa_1b_1^2\\&-3k^2a_0^2a_{-1}-3k^2a_1a_{-1}^2+3cka_0^2b_{-1}-2c^2a_1b_{-1}^2\\&+6cka_0a_{-1}b_0+2k^4b^2a_{-1}b_0^2-4c^2a_0b_{-1}b_0\\&+2k^4aa_{-1}b_0^2-2c^2a_{-1}b_0^2=0,\end{aligned}$$

$$\begin{aligned}e^{0\xi}:\ &-4c^2a_0b_1b_{-1}+18k^4aa_2b_{-1}^2-18aa_{-1}b_{-1}\\&-4c^2a_0b_1b_{-1}+18k^4aa_2b_{-1}^2-18k^4aa_{-1}b_{-1}-k^2a_0^3\\&+18k^4b^2a_2b_{-1}^2-18k^4b^2a_{-1}b_{-1}+3cka_{-1}^2-2c^2a_0b_0^2\\&12k^4aa_0b_{-1}b_{-1}+6cka_1a_0b_{-1}+6cka_0a_{-1}b_1\\&6k^4aa_{-1}b_0b_{-1}+6k^4aa_{-1}b_1b_0+6k^4b^2a_1b_0b_{-1}\\&-6k^4b^2a_{1-}b_1b_0-6ckb^2a_1a_{-1}b_0+4c^2a_{-1}b_{-1}\\&+2c^2a_2b_{-1}^2-3k^2a_2a_{-1}^2+3cka_0^2b_0=0,\end{aligned}$$

$$\begin{aligned}e^{\xi}:\ &6cka_0a_{-1}-26k^4b^2a_0b_{-1}+8k^4b^2a_{-1}b_1^2-3k^2a_1a_0^2\\&+3cka_1^2b_{-1}-4c^2a_1b_1b_{-1}-6k^2a_2a_0a_{-1}2k^4aa_1b_0^2\\&+4k^4aa_{-1}b_0+4k^4b^2a_{-1}b_0-4c^2a_2b_0b_{-1}+\\&+2k^4b^2a_1b_0^2-4c^2a_0b_1b_{-1}+3cka_0^2b_1+8k^4aa_{-1}b_1^2\\&-26k^4aa_0b_{-1}-2c^2a_{-1}b_1^2-3k^2a_1^2a_{-1}-2c^2a_1b_0^2\\&-4c^2a_0b_{-1}-4c^2a_{-1}b_0-8k^4aa_1b_1b_{-1}+22k^4aa_2b_0\\&-8k^4b^2aa_1b_1b_{-1}+6cka_1a_{-1}b_1+6cka_1a_0b_{-1}\\&-2k^4aa_0b_1b_0-2k^4b^2a_0b_1b_0+6cka_1a_0b_0\\&+22k^4b^2a_2b_0b_{-1}+6cka_2a_{-1}b_0=0,\end{aligned}$$

$$\begin{aligned}e^{2\xi}:\ &-26k^4b^2a_1b_{-1}+22k^4b^2a_{-1}b_1-6k^2a_2a_1a_{-1}\\&+6cka_1a_{-1}-4c^2a_2b_1b_{-1}+3cka_1^2b_0+8k^4aa_2b_0^2\\&+8k^4aa_0b_0+8k^4b^2a_2b_0^2-8k^4b^2a_0b_0-4c^2a_1b_1b_0\\&+2k^2aa_0b_1^2+2k^4b^2a_0b_0-26k^4aa_1b_{-1}-2c^2a_2b_0^2\\&+22k^4aa_{-1}b_1-3k^2a_2a_0^2-3k^2a_1^2a_0-4c^2a_1b_{-1}\\&-2c^2a_0b_1^2-4c^2a_0b_0-4c^2a_{-1}b_1+3cka_0^2\\&-2k^4b^2a_2b_1b_0+6cka_2a_{-1}b_1+6cka_2a_1b_{-1}\\&+6cka_1a_0b_1-2k^4aa_1b_1b_0=0,\end{aligned}$$

$$\begin{aligned}e^{3\xi}:\ &6cka_2a_1b_0+6k^4aa_2b_1b_0+6k^4b^2a_2b_1b_0-4c^2a_1b_0\\&-18k^4b^2a_2b_{-1}+3cka_2^2b_{-1}+6cka_2a_{-1}-12k^4aa_1b_0\\&-12k^4b^2a_1b_0-4c^2a_2b_1b_0+3cka_1^2b_1+6k^4aa_0b_1\\&+6k^4b^2a_0b_1-6k^2a_2a_1a_0+6cka_1a_0-18k^4aa_2b_{-1}\\&-k^2a_1^3-2c^2a_{-1}-4c^2a_0b_1-2c^2a_1b_1^2+18k^4b^2a_{-1}\\&+18k^4aa_{-1}-4c^2a_2b_1-3k^2a_2^2a_{-1}=0,\end{aligned}$$

$$\begin{aligned}e^{4\xi}:\ &8k^4aa_0+3cka_2^2b_0+8k^4b^2a_0-8k^4b^2a_2b_0+3cka_1^2\\&+6cka_2a_1b_1-2c^2a_2b_1^2-4c^2a_1b_1-4c^2a_2b_0+\end{aligned}$$

$$+2k^4aa_2b_1^2-2k^4aa_1b_1-8k^4aa_2b_0+2k^4b^2a_2b_1^2$$
$$-2k^4b^2a_1b_1-3k^2a_2^2a_0-3k^2a_2a_1^2+6cka_2a_0$$
$$-2c^2a_0=0,$$

$$e^{5\xi}:-4c^2a_2b_1+2k^4b^2a_1-2k^4b^2a_2b_1-3k^2a_2^2a_1$$
$$+k^4aa_1-2k^4aa_2b_1+3cka_2^2b_1+6cka_2a_1-2c^2a_1$$
$$-2c^2a_1=0,$$

$$e^{6\xi}:3cka_2^2-k^2a_2^3-2c^2a_2=0.$$

Solving these equations with the aid of Maple we get the nine sets of solutions as follows:

1. $a_{-1}=0, a_0=0, a_1=0, a_2=\pm 6k\sqrt{a+b^2},$

$b_{-1}=b_{-1}, b_0=0, b_1=0,$

$$c=\pm 3k^2\sqrt{a+b^2}, k=k. \quad (36)$$

2. $a_{-1}=0, a_0=0, a_1=0, a_2=\pm 4k\sqrt{a+b^2},$

$b_{-1}=0, b_0=b_0, b_1=0,$

$$c=\pm 2k^2\sqrt{a+b^2}, k=k. \quad (37)$$

3. $a_{-1}=0, a_0=0, a_1=\pm 2b_1k\sqrt{a+b^2}, a_2=0,$

$b_{-1}=0, b_0=0, b_1=b_1,$

$$c=\pm k^2\sqrt{a+b^2}, k=k. \quad (38)$$

4. $a_{-1}=0, a_0=0, a_1=\pm 4b_1k\sqrt{a+b^2},$

$a_2=\pm 4b_1k\sqrt{a+b^2}, b_{-1}=b_0b_10,$

$$b_0=b_0, b_1=b_1, c=\pm 2k^2\sqrt{a+b^2}, k=k. \quad (39)$$

5. $a_{-1}=\pm 4b_0b_1k\sqrt{a+b^2}, a_0=\pm b_0k\sqrt{a+b^2},$

$a_1=0, a_2=0, b_{-1}=b_0b_1, b_0=b_0,$

$$b_1=b_1, c=\pm 2k^2\sqrt{a+b^2}, k=k. \quad (40)$$

6. $a_{-1}=0, a_0=\pm 2b_0k\sqrt{a+b^2}, a_1=a_1,$

$a_2=0, b_{-1}=0, b_0=b_0,$

$b_1=\frac{\pm ka_1(a_1^2+4b_0k^2b^2+4b_0k^2a)\sqrt{a+b^2}}{2(a+b^2)}0,$

$$c=\pm k^2\sqrt{a+b^2}, k=k. \quad (41)$$

7. $a_{-1}=\pm 6b_{-1}k\sqrt{a+b^2}, , a_0=0, a_1=0,$

$a_2=0, b_{-1}=b_{-1}, b_0=0, b_1=0,$

$$c=\pm 3k^2\sqrt{a+b^2}, k=k. \quad (42)$$

8. $a_{-1}=0, a_0=\pm 4b_0k\sqrt{a+b^2}, a_1=0,$

$a_2=0, b_{-1}=0, b_0=b_0, b_1=0,$

$$c=\pm 2k^2\sqrt{a+b^2}, k=k. \quad (43)$$

9. $a_{-1}=a_{-1}, a_0=a_0, a_1=a_1, a_2=0,$

$b_{-1}=\frac{\pm a_{-1}}{2k\sqrt{a+b^2}}, b_0=\frac{a_{-1}}{a_1}+\frac{a_0}{2k\sqrt{a+b^2}},$

$$b_1=\frac{a_0}{a_1}+\frac{a_1}{2k\sqrt{a+b^2}}, c=\pm k^2\sqrt{a+b^2}, k=k. \quad (44)$$

Substituting Eqs. (36) - (44) into Eq. (35) we obtain respectively the following solutions:

$$u_1(\xi)=\frac{6k\sqrt{a+b^2}e^{2\xi}}{e^{2\xi}+b_{-1}e^{-\xi}},$$

$$v_1(\xi)=\frac{18k^2e^{\xi}b_{-1}\left(a+b^2-b\sqrt{a+b^2}\right)}{\left(e^{2\xi}+b_{-1}e^{-\xi}\right)^2}. \quad (45)$$

$$u_2(\xi)=\frac{4k\sqrt{a+b^2}e^{2\xi}}{e^{2\xi}+b_0},$$

$$v_2(\xi)=\frac{8b_0k^2e^{2\xi}b_{-1}\left(a+b^2-b\sqrt{a+b^2}\right)}{\left(e^{2\xi}+b_0\right)^2}. \quad (46)$$

$$u_3(\xi)=\frac{2b_1k\sqrt{a+b^2}e^{\xi}}{e^{2\xi}+b_1e^{\xi}},$$

$$v_3(\xi)=\frac{2b_1k^2e^{\xi}\left(a+b^2+b\sqrt{a+b^2}\right)}{\left(e^{\xi}+b_1\right)^2}. \quad (47)$$

$$u_4(\xi)=\frac{1}{\psi}\left(4k\sqrt{a+b^2}(e^{\xi}+b_1)e^{\xi}\right),$$

$$v_4(\xi)=\frac{8k^2b_0}{\psi^2}\Big[(bb_1^2\sqrt{a+b^2}-a-b^2)$$
$$+(2b_1+2b_1b^2-2b\sqrt{a+b^2})e^{\xi}$$
$$+(a+b^2-b\sqrt{a+b^2})e^{2\xi}\Big]. \quad (48)$$

$$u_5(\xi)=\frac{4b_0k\sqrt{a+b^2}(1+b_1e^{\xi})}{\psi},$$

$$v_5(\xi) = \frac{8k^2 b_0}{\psi^2}\left[(2ab_1 + 2b^2 b_1 + 2bb_1\sqrt{a+b^2})e^{\xi}\right.$$
$$+(a + b^2 + b\sqrt{a+b^2})e^{2\xi}$$
$$\left.+ab_1^2 b^2 b_1^2 + bb_1^2\sqrt{a+b^2}\right]. \quad (49)$$

Where

$$\psi = e^{2\xi} + b_1 e^{\xi} + b_0 + b_1 b_0 e^{-\xi}.$$

$$u_6(\xi) = \frac{1}{\omega}\left(a_1 e^{\xi} + 2b_0 k\sqrt{a+b^2}\right),$$

$$v_6(\xi) = \frac{1}{\omega}\left((k\sqrt{a+b^2}e^{\xi} + 2b_0 k(a+b^2)\right).$$
$$-\frac{1}{2\omega^2}\left(a_1 e^{\xi} + 2b_0 k\sqrt{a+b^2}\right)^2 - \frac{bk}{\omega}a_1 e^{\xi}$$
$$+\frac{bk}{\omega^2}\left(a_1 e^{\xi} + 2b_0 k\sqrt{a+b^2}\right).$$
$$\times\left(e^{2\xi} + \frac{(a_1^2 + 4k^2 b_0 a + 4k^2 b_0 b^2)e^{\xi}}{2ka_1\sqrt{a+b^2}}\right). \quad (50)$$

Where

$$\omega = e^{2\xi} + \frac{(a_1^2 + 4k^2 b_0 a + 4k^2 b_0 b^2)e^{\xi}}{2ka_1\sqrt{a+b^2}} + b_0.$$

$$u_7(\xi) = \frac{6kb_{-1}\sqrt{a+b^2}e^{-\xi}}{e^{2\xi} + b_{-1}e^{-\xi}},$$

$$v_7(\xi) = \frac{18k^2 b_{-1}(a + b^2 + b\sqrt{a+b^2})e^{\xi}}{(e^{2\xi} + b_{-1}e^{-\xi})^2}. \quad (51)$$

$$u_8(\xi) = \frac{4b_0 k\sqrt{a+b^2}}{e^{2\xi} + b_0},$$
$$v_8(\xi) = \frac{8k^2 b_0(a + b^2 + b\sqrt{a+b^2})e^{2\xi}}{(e^{2\xi} + b_0)^2}. \quad (52)$$

$$u_9(\xi) = \frac{2a_1 k}{\Gamma}\left((a_1 e^{\xi} + a_0 + a_{-1}e^{-\xi})\sqrt{a+b^2}\right.$$
$$v_9(\xi) = \frac{2a_1 k^2}{\Gamma}\left((a_1 e^{\xi} + a_0 + a_{-1}e^{-\xi})(a+b^2),\right.$$
$$-\frac{2a_1^2 k^2}{\Gamma^2}\left(a+b^2\right)\left(a_1 e^{\xi} + a_0 + a_{-1}e^{-\xi}\right)^2$$
$$-\frac{2a_1 bk^2}{\Gamma}\left(a_1 e^{\xi} + a_{-1}e^{-\xi}\right)\sqrt{a+b^2})$$
$$+\frac{2bk^2 a_1}{\Gamma^2}\left[(a_1 e^{\xi} + a_0 + a_{-1}e^{-1})\right.$$
$$\times\left(4a_1 e^{2\xi}(a+b^2) + 2a_0 k(a+b^2)e^{\xi}\right.$$
$$+\left(a_1^2 e^{\xi} - a_{-1}a_1 e^{-\xi}\right) \quad (53)$$

Where

$$\Gamma = aa_{-1}e^{-\xi} + \left(a_0 a_1 + 2a_{-1}k\sqrt{a+b^2}\right))$$
$$+\left(a_1^2 + 2a_0 k\sqrt{a+b^2}\right)e^{\xi} + 2a_1 k\sqrt{a+b^2}e^{2\xi},$$

$$u(\xi) = u(x,t), \quad \xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}.$$

Case 3: For the case with p = c = 2, q = d =2, Eq. (18) becomes

$$u(\xi) = \frac{a_2 e^{2\xi} + a_1 e^{\xi} + a_0 + a_{-1}e^{-\xi} + a_{-2}e^{-2\xi}}{e^{2\xi} + b_1 e^{\xi} + b_0 + b_{-2}e^{-2\xi} \ldots b_{-1}e^{-\xi}} \quad (54)$$

For simplify, we take $b_{-1} = 0$, $b_1 = 0$. Then Eq.(54) becomes

$$u(\xi) = \frac{a_2 e^{2\xi} + a_1 e^{\xi} + a_0 + a_{-1}e^{-\xi} + a_{-2}e^{-2\xi}}{e^{2\xi} + b_0 + b_{-2}e^{-2\xi}} \quad (55)$$

Substituting Eq. (55) into Eq. (17) and taking the coefficients of e^{ξ} in each term zero yields to a set of algebraic equations for a_{-2}, a_{-1}, a_0, a_1, a_2, b_0, b_{-2}, c, and k as follows

$$e^{-6\xi}: -2c^2 a_{-2}b_{-2}^2 + 3cka_{-2}^2 b_{-2} - k^2 a_{-2}^3 = 0,$$

$$e^{-5\xi}: -3k^2 a_{-2}^2 a_{-1} + 6cka_{-2}a_{-1}b_{-2} + 2k^4 aa_{-1}b_{-2}^2$$
$$-2c^2 a_{-1}b_{-2}^2 + 2k^4 b^2 a_{-1}b_{-2}^2 = 0,$$

$$e^{-4\xi}: -2c^2 a_0 b_{-2}^2 + 6cka_0 a_{-2}b_{-2} + 3cka_{-2}^2 b_0$$
$$+3cka_{-2}^2 b_{-2} + 8k^4 b^2 a_0 b_{-2}^2 - 8k^4 b^2 a_0 b_0 b_{-2}$$
$$-8k^4 aa_0 b_{-2}^2 - 8k^4 aa_{-2}b_0 b_{-2} - 4c^2 a_{-2}b_0 b_2$$
$$-3k^2 a_0 a_{-2}^2 = 0,$$

$$e^{-3\xi}: 18k^4 b^2 a_1 b_{-2}^2 + 6cka_{-2}a_{-1}b_0 + 3k^2 a_1 a_{-2}^2$$
$$-4c^2 a_{-1}b_0 b_{-2} - 2c^2 a_1 b_{-2}^2 + 18k^4 aa_1 b_{-2}^2$$
$$+6cka_1 a_{-2}b_{-2} - 12k^4 aa_{-1}b_0 b_{-2} - 6k^2 a_0 a_{-2}a_{-1}$$
$$-12k^4 b^2 a_{-1}b_0 b_{-2} - k^2 a_{-1}^3 + 6cka_0 a_{-1}b_{-2} = 0,$$

$$e^{-2\xi}: -32k^4 aa_2 b_{-2} + 6cka_1 a_{-1} + 6cka_2 a_0 b_0 - 4c^2 a_0 b_0$$
$$-32k^4 b^2 a_2 b_{-2} + 3cka_1^2 b_0 + 32k^4 b^2 a_{-2} + 3cka_0^2$$
$$+6cka_2 a_{-2} + 32k^4 aa_{-2} - 4c^2 a_2 b_{-2} - 6k^2 a_2 a_1 a_{-1}$$
$$-2c^2 a_{-2} + 3cka_2^2 b_{-2} - 3k^2 a_2 a_0^2 + 8k^4 aa_2 b_0^2$$
$$+8k^4 b^2 a_2 b_0^2 - 8k^4 aa_0 b_0 - 3k^2 a_1^2 a_0 - 2c^2 a_2 b_0^2$$
$$+8k^4 b^2 a_0 b_0 = 0,$$

$$\mathbf{e^{-\xi}}: -6k^2a_1a_0a_{-2} + 6cka_1a_{-2}b_0 - 4c^2a_1b_0b_{-2} + 6cka_1a_0b_{-2} - 6k^2a_2a_{-2}a_{-1} + 4k^4aa_1b_0b_{-2} - 3k^2a_1a_{-1}^2 - 3k^2a_0^2a_{-1} + 6cka_2a_{-2}b_{-2} - 2c^2a_{-1}b_0^2 + 6cka_{-2}a_{-1} + 2k^4b^2a_{-1}b_0^2 + 2k^4aa_{-1}b_0^2 - 44k^4b^2a_{-1}b_{-2} + 4k^4b^2a_1b_0b_{-2} + 6cka_0a_{-1}b_0 - 4c^2a_{-1}b_{-2} - 44k^4aa_{-1}b_{-2} = 0,$$

$$\mathbf{e^{0\xi}}: 3cka_{-1}^2 - 3k^2a_2a_{-1}^2 - k^2a_0^3 - 4c^2a_0b_{-2} - 4c^2a_2b_0b_{-2} - 6cka_2a_0b_{-2} - 24k^4b^2a_2b_0b_{-2} + 24k^4aa_2b_0b_{-2} + 6cka_2a_{-2}b_0 + 6cka_1a_{-1}b_0 + 3cka_0^2b_0 - 48k^4aa_0b_{-2} + 6cka_0a_{-2} - 4c^2a_{-2}b_0 - 6k^2a_2a_0a_{-2} - 6k^2a_1a_0a_{-1} - 48k^4b^2a_0b_{-2} + 24k^4b^2a_{-2}b_0 + 24k^4aa_{-2}b_0 = 0,$$

$$\mathbf{e^{\xi}}: 6cka_1a_0b_0 - 6k^2a_2a_1a_{-2} - 44k^4aa_1b_{-2} + 2k^4b^2a_1b_0^2 \; 2k^4aa_1b_0^2 + 6cka_0a_{-1} + 4k^4b^2a_{1-}b_0 - 4c^2a_{-1}b_0 + 6cka_2a_1b_{-2} - 6cka_2a_{-1}b_0 - 6k^2a_2a_0a_{-1} - 2c^2a_1b_0^2 - 3k^2a_1a_0^2 - 4c^2a_1b_{-2} + 6cka_1a_{-2} - 3k^2a_1^2a_{-1} - 44k^4b^2a_1b_{-2} + 4k^4aa_{-1}b_0 = 0,$$

$$\mathbf{e^{2\xi}}: -6k^2a_1a_{-2}a_{-1} + 6cka_2a_{-2}b_{-2} - 32k^4b^2a_{-2}b_{-2} + 3cka_0^2b_{-2} - 2c^2a_{-1}b_0^2 + 6cka_1a_{-1}b_{-2} - 2c^2a_2b_{-2}^2 - 3k^2a_0^2a_{-2} - 4c^2a_{-2}b_{-2} + 3cka_{-1}^2b_0 + 3cka_{-2}^2 - 8k^4aa_0b_{-2}b_0 + 8k^4aa_{-2}b_0^2 - 8k^4b^2a_0b_{-2}b_0 - 3k^2a_2a_{-2}^2 + 6cka_0a_{-2}b_0 - 32k^2aa_{-2}b_{-2} + 8k^4b^2a_{-2}b_0^2 + 32k^4aa_2b_{-2}^2 - 4c^2a_0b_{-2}b_0 - 3k^2a_0a_{-1}^2 + 32k^4b^2a_2b_{-2}^2 = 0,$$

$$\mathbf{e^{3\xi}}: -4c^2a_1b_0 - 2c^2a_{-1} + 6cka_2a_{-1} + 18k^4aa_{-1} - 12k^4aa_1b_0 - 3k^2a_2^2a_{-1} - 12k^4b^2a_1a_{-1} - k^2a_1^3 + 6cka_2a_1b_0 + 18k^4b^2a_{-1} - 6k^2a_2a_1a_0 + 6cka_1a_0 = 0,$$

$$\mathbf{e^{4\xi}}: 3cka_1^2 + 8k^4aa_0 + 3ckb_2^2b_0 - 3k^2a_2a_1^2 - 2c^2a_0 + 8k^4b^2a_0 - 8k^4aa_2b_0 - 8k^4b^2a_2b_0 - 4c^2a_2b_0 + 6cka_2b_0 - 3k^2a_2^2a_0 = 0,$$

$$\mathbf{e^{5\xi}}: 6cka_2a_1 + 2k^4b^2a_1 - 2c^2a_1 - 3k^2a_2^2a_1 + 2k^4aa_1 = 0,$$

$$\mathbf{e^{6\xi}}: -k^2a_2^3 - 2c^2a_2 + 3cka_2^2 = 0.$$

Solving these equations with the aid of Maple we get the five sets of solutions as follows:

1. $a_{-2} = 0, a_{-1} = 0, a_0 = 0, a_1 = 0,$

$$a_2 = \pm 8k\sqrt{a+b^2}, b_{-2} = b_{-2}, b_0 = 0,$$

$$c = \pm 4k^2\sqrt{a+b^2}, k = k. \tag{56}$$

2. $a_{-2} = 0, a_{-1} = 0, a_0 = \pm 4b_0k\sqrt{a+b^2},$

$$a_1 = 0, a_2 = 0, b_{-2} = 0, b_0 = b_0,$$

$$c = \pm 2k^2\sqrt{a+b^2}, k = k. \tag{57}$$

3. $a_{-2} = 0, a_{-1} = 0, a_0 = a_0, a_1 = 0,$

$$a_2 = \pm 4\sqrt{a+b^2}, b_{-2} = \frac{\pm 4a_0b_0k\sqrt{a+b^2}}{16k^2(a+b^2)},$$

$$b_0 = b_0, c = \pm 2\sqrt{a+b^2}, k = k. \tag{58}$$

4. $a_{-2} = 0, a_{-1} = 0, a_0 = \pm 2b_0k\sqrt{a+b^2},$

$$a_1 = \pm 2ik\sqrt{a+b^2}, a_2 = 0, b_{-2} = 0,$$

$$b_0 = b_0, c = \pm k^2\sqrt{a+b^2}, k = k. \tag{59}$$

5. $a_{-2} = \pm 8k\sqrt{a+b^2}, a_{-1} = 0, a_0 = 0,$

$$a_1 = 0, a_2 = 0, b_{-2} = b_{-2}, b_0 = 0,$$

$$c = \pm 4k^2\sqrt{a+b^2}, k = k. \tag{60}$$

Substituting Eqs. (56) - (60) into Eq. (55) we obtain respectively the following solutions:

$$u_1(\xi) = \frac{8k\sqrt{a+b^2}e^{2\xi}}{e^{2\xi}+b_{-2}e^{-2\xi}},$$

$$v_1(\xi) = \frac{32k^2b_{-2}(a+b^2-b\sqrt{a+b^2})}{(e^{2\xi}+b_{-2}e^{-2\xi})^2}. \tag{61}$$

$$u_2(\xi) = \frac{4b_0k\sqrt{a+b^2}}{e^{2\xi}+b_0},$$

$$v_2(\xi) = \frac{8k^2b_0(a+b^2+b\sqrt{a+b^2})e^{2\xi}}{(e^{2\xi}+b_0)^2}. \tag{62}$$

$$u_3(\xi) = \frac{4k\sqrt{a+b^2}e^{2\xi}+a_0}{e^{2\xi}+b_0+\frac{(4a_0b_0k\sqrt{a+b^2}-a_0^2)e^{-2\xi}}{16k^2(a+b^2)}},$$

$$v_3(\xi) = \frac{-1}{\Delta}\left(2ka_0\sqrt{a+b^2} + 8k^2(a+b^2)\right)$$

$$-\frac{1}{2\Delta^2}\left(4k\sqrt{a+b^2}+a_0\right)^2-\frac{8b}{\Delta^2}\left((\sqrt{a+b^2}e^{2\xi}\right)$$

$$+\frac{bk}{\Delta^2}\left(8k\sqrt{a+b^2}e^{2\xi}\right)+\frac{bk}{\Delta^2}\left(4k\sqrt{a+b^2}e^{2\eta}+a_0\right)$$

$$\times\left(2e^{2\xi}-\frac{(4a_0b_0k\sqrt{a+b^2}-a_0^2)(e^{-2\xi})}{8k^2(a+b^2)}\right). \qquad (63)$$

Where

$$\Delta=e^{2\xi}+b_0+\frac{(4a_0b_0k\sqrt{a+b^2}-a_0^2)e^{-2\xi}}{16k^2(a+b^2)}.$$

$$u_4(\xi)=\frac{\left(2k(i\sqrt{b_0}e^{\xi}+b_0)\sqrt{a+b^2}\right)}{e^{2\xi}+b_0},$$

$$v_4(\xi)=\frac{2k^2}{K}\left((a+b^2)(i\sqrt{b_0}e^{2\xi}+b_0)\right)$$

$$-\frac{2k^2}{K^2}(a+b^2)(i\sqrt{b_0}e^{\xi}+b_0)^2$$

$$\frac{-2ibk^2}{K}\left(\sqrt{b_0(a+b^2)}e^{2\xi}\right)$$

$$+\frac{4bk^2}{K}\left((i\sqrt{b_0}e^{2\xi}+b_0)\sqrt{a+b^2}e^{2\xi}\right). \qquad (64)$$

Where

$$K=e^{2\xi}+b_0.$$

$$u_5(\xi)=\frac{8kb_{-1}\sqrt{a+b^2}e^{-2\xi}}{e^{2\xi}+b_{-2}e^{-2\xi}},$$

$$v_5(\xi)=\frac{1}{\Lambda}\left(32b_{-2}k^2(a+b^2)e^{-2\xi}\right)$$

$$-\frac{32k^2b_{-2}^2}{\Lambda^2}(a+b^2)\left(e^{-2\xi}\right)^2$$

$$+\frac{16bk^2b_{-2}}{\Lambda}\left(\sqrt{a+b^2}e^{-2\xi}\right)$$

$$+\frac{8bk^2b_{-2}}{\Lambda^2}\sqrt{a+b^2}e^{-2\xi}$$

$$\times\left(2e^{-2\xi}-2b_{-2}e^{-2\xi}\right). \qquad (65)$$

Where

$$\Lambda=e^{2\xi}+b_{-2}e^{-2\xi}, u(\xi)=u(x,t),\ \xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}.$$

V. Conclusions

In this paper, we successfully use the Exp-function method to solve fractional nonlinear partial differential equations with Jumarie's modified Riemann–Liouville derivative. This method is reliable and efficient. To our knowledge, the solutions obtained in this paper have not been reported in the literature so far.

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