Adomian Decomposition Method
For solving Fractional Differential Equations

Mahmoud M. El-Borai, Wagdy G. El-Sayed, Adham M. Jawad

Department of Mathematics, Faculty of Science, Alexandria University , Alexandria – Egypt

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Abstract: The Adomian decomposition method (ADM) is a non numerical method for solving a wide variety of functional equations and usually gets the solution in a series form.
System of fractional partial differential equation which has numerous applications in many fields of science is considered. Adomian decomposition method, a novel method is used to solve these type of equations. The solutions are derived in convergent series form which shows the effectiveness of the method for solving a wide variety of fractional differential equations.

Keywords: Adomian decomposition method, Fractional partial differential equations, System of differential equations, initial value problems.

Introduction

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in many fields of science and engineering[9,10]. One of these applications, Adomian decomposition method (ADM)introduced by Adomian (1980), provides an effective procedure for finding explicit and numerical solutions of a wider and general class of differential systems representing real physical problems [2,3]. This method efficiently works for initial value or boundary value problems, for linear or nonlinear, ordinary or partial differential equations[5,6], and even for stochastic systems as well. Moreover, no linearization or perturbation is required in this method.

2. PRELIMINARIES AND NOTATIONS:

If

$f(t)$ *is continous on an interval* $[a, b]$ and $0 < \alpha \leq 1$

,then the operator $I_{0^+}^{\alpha}$, defined by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t)}{(t-s)^{1-\alpha}} ds \qquad (1)$$

is called the Riemann- Liouville fractional integral operator of order α. Here $\Gamma(.)$ is the Gamma function .

The Caputo time fractional derivative of order $\alpha > 0$ [16], is defined as:

$$D_t^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,t)}{\partial t^m}, & (m-1 < \alpha < m) \\ \frac{\partial^m u(x,\tau)}{\partial t^m}, & (m = \alpha \in N) \end{cases}$$

.... **(2)**

Lemma 1(See [1])

Let $\rho, q \geq 0$, $f(t) \in L_1[0,T]$. Then $I_{0^+}^{\rho} I_{0^+}^{q} f(t) = I_{0^+}^{\rho+q} f(t) = I_{0^+}^{q} I_{0^+}^{\rho} f(t)$ is satisfied almost everywhere on$[0,T]$. Moreover, if $f(t) \in L_1[0,T]$, then the above equation is true for all $t \in [0,T]$.

Lemma2 (See [1])

If $\alpha > 0, f(t) \in L_1[0,T]$, then $^{c}D^{\alpha}_{0^+} I^{\alpha}_{0^+} f(t) = f(t)$, for all $t \in [0,T]$

Basic idea of Adomian Decomposition Method

Consider the differential equation.

$$Lu + Ru + Nu = g \qquad (3)$$

Where

L – Highest order derivative and easily invertible.

R – Linear differential operator of order less than L,

g – Source term.

Nu –Represents the nonlinear terms. The function, u(t) is assumed to be bounded for all t∈ I=[0,T] and the nonlinear term Nu satisfies Lipschitz condition i.e. $|Nu\text{-}Nv| \leq L_1 |u\text{-}v|$. Where L_1 is a positive constant.

Because L is invertible, we get

$$u = \phi + L^{-1}g - L^{-1}Ru - L^{-1}Nu \qquad (4)$$

Where ϕ is the integration constant and satisfies Lϕ = 0 and

$$L^{-1}(.) = \int_0^t (.)dt$$

The unkown function u is given by the infinite series,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (5)$$

And the nonlinear term Nuwill be decomposed by the infinite series of Adomian polynomials

$$Nu = \sum_{n=0}^{\infty} A_n\,(u_0, u_1, \dots, u_n) \qquad (6)$$

Where A_n is Adomian polynomial calculated by using the formula

$$An = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(v(\lambda)\right)\right]_{\lambda=0},$$

$n = 0,1,2,\dots$

Where

$$v(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n$$

Substituting the decomposition series Eq. (5) and Eq. (6) into Eq. (4), gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \varphi + L^{-1}g - L^{-1}R\left(\sum_{n=0}^{\infty} u_n(x,t)\right) - \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n) \qquad (7)$$

From the above equation, we observe that

$$u_0 = \varphi + L^{-1}g$$

$$u_1 = -L^{-1}(Ru_0) - L^{-1}(A_0)$$

$$u_2 = -L^{-1}(Ru_1) - L^{-1}(A_1)$$

$$\vdots$$

$$u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n) \quad , \quad n \geq 0$$

Where φ is the initial condition.

Hence all terms of u are calculated and the general solution obtained according to ADM as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

The convergence of this series has been proved [8].

Now, we apply Adomian decomposition method to derive the solution of fractional partial differential equations.We solve five examples by Adomian Decomposition Method.

Firstly, we apply the Adomian decomposition method to obtain approximate solutions of IVPs for fractional BBM-Burger's equation with $\mathcal{E} = 1$

Example 1.

Consider now the following equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \left(L_{xx}u(x,t)\right) - \left(u(x,t)L_x u(x,t)\right)$$

Where $L_{xx} = \frac{\partial^2}{\partial x^2}$, $L_x = \frac{\partial}{\partial x}$

With the initial condition

$u(x,0) = \sin x\,, \qquad (x,t) \in [0,1] \times (0,T]$

And the fractional differential operator $\frac{\partial^\alpha}{\partial t^\alpha}$ defined by (2).Let J^αbe the inverse of the operator $\frac{\partial^\alpha}{\partial t^\alpha}$, now applying J^α to the both sides of (10), we get

$$u(x,t) = \phi + J^\alpha(L_{xx}u) - J^\alpha(Nu)$$

where$Nu = u\,L_x u$

In order to solve our problem,we must generalize these Adomian polynomials in follows.

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\frac{\partial}{\partial x}\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, n \geq 0$$

The first few terms of the Adomian polynomials are derived as follows

$$A_0 = u_0\frac{\partial u_0}{\partial x}$$

$$A_1 = \frac{1}{1!}\left[\frac{d}{d\lambda}\left[(u_0 + \lambda u_1)\left(\frac{\partial u_0}{\partial x} + \lambda\frac{\partial u_1}{\partial x}\right)\right]\right]_{\lambda=0}$$

$$= u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x_1}$$

$$A_2 = \frac{1}{2!}\left[\frac{d^2}{d\lambda^2}\left[(u_0 + \lambda u_1 + \lambda^2 u^2)\left(\frac{\partial u_0}{\partial x} + \lambda\frac{\partial u_1}{\partial x} + \lambda^2\frac{\partial u_2}{\partial x}\right)\right]\right]_{\lambda=0}$$

$$= u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x}$$

Similarly

$$A_3 = u_0\frac{\partial u_3}{\partial x} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} + u_3\frac{\partial u_0}{\partial x}$$

$\vdots$

And so on

From (7), we get

$$u(x,t) = \varphi + J^\alpha\left(\sum_{n=0}^{\infty}(L_{xx}u_n)\right) - \; - \; J^\alpha\left(\sum_{n=0}^{\infty}(A_n)\right)$$

,

$$u_0 = \varphi = u(x,0)$$

$$u_1 = J^\alpha(L_{xx}u_0) - J^\alpha(A_0)$$

$$u_2 = J^\alpha(L_{xx}u_1) - J^\alpha(A_1)$$

$\vdots$

$$u_{n+1} = J^\alpha(L_{xx}u_n) - J^\alpha(A_n)$$

By substituting the values of$u_0, u_1, \ldots$ from above, we get the solution of the IVP

$$u(x,t) = u_0 + u_1 + \cdots + u_n + \cdots$$

$$u_0 = u(x,0) = f(x) = \sin x$$

$$u_1 = J^\alpha(L_{xx}u_0) - J^\alpha(A_0)$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-\theta)^{1-\alpha}} f''(x)d\theta - \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-\theta)^{1-\alpha}}[f(x$$

$$u_1 == \frac{f_1(x)}{\Gamma(\alpha+1)}t^\alpha$$

Where $f_1(x) = -f''(x) + f(x)f'(x)$

$= \sin(x)(1 + \cos x).$

$$u_2 = J^\alpha(L_{xx}u_1) - J^\alpha(A_1)$$

$$= f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Where $f_2(x) = f_1''(x) - f(x)f_1'(x) - f_1(x)f'(x)$

$= \sin^3(x) + [-1 - 5\cos x - \cos^2 x - (1 + \cos x)\cos x]\sin x$

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$$u_3 = J^\alpha(A_2) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-\theta)^{1-\alpha}}\left[u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x}\right]d\theta$$

$$u_3 = J^\alpha(L_{xx}u_2) - J^\alpha(A_2)$$

$$= f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

Similarly

$$u_4 = \frac{f_4(x)}{\Gamma(4\alpha+1)}t^{4\alpha}$$

$\vdots$

$$u_n = \frac{f_n(x)}{\Gamma(n\alpha+1)}t^{n\alpha}$$

The solution of the equation in series is given by

$$u(x,t) = f(x) + \frac{t^\alpha}{\Gamma(\alpha+1)}f_1(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}f_2(x) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}f_3(x) \ldots + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}f_n(x) + \cdots$$

$= \sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}f_n(x)$, Where $f_0(x)$ is an initial condition.

Next, we will solve a more general system of nonlinear fractional differential equations.

Example 2.

Consider the following of nonlinear fractional differential equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = 0$$

$(x,t) \in \Omega \times (0,T]$ and $0 < \alpha \leq 1$

With the initial condition $u(x,0) = f(x)$

Where $f(x) = cosx$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$$

$$= -u(x,t)L_x u(x,t) + L_{xxx}u(x,t) - L_{xx}u(x,t) + L_x$$

Where $L_{xxx} = \frac{\partial^3}{\partial x^3}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$, $L_x = \frac{\partial}{\partial x}$

And the fractional D_t^α defined as Eq. (2) we know that J^α is inverse of the operator D_t^α

Now applying J^α to the both side of the given Eq., we obtain.

$$u(x,t) = \phi - J^\alpha(Nu) + J^\alpha(L_{xxx}u) - J^\alpha(L_{xx}u) + J^\alpha(L_x u)$$

Where $Nu = u\frac{\partial u}{\partial x}$

The first few terms of the Adomian polynomials are given by:

$$A_0 = u_0\frac{\partial u_0}{\partial x}$$

$$A_1 = u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}$$

$$A_2 = u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u}{\partial x} + u_2\frac{\partial u_0}{\partial x}$$

$\vdots$

And so on

From (7), we get

$$\sum_{n=0}^{\infty}u_n(x,t) = \varphi - J^\alpha\left(\sum_{n=0}^{\infty}A_n(u_0,u_1,\ldots,u_n)\right) + J^\alpha\left(\sum_{n=0}^{\infty}L_{xxx}u_n\right) - J^\alpha\left(\sum_{n=0}^{\infty}L_{xx}u_n\right) + J^\alpha\left(\sum_{n=0}^{\infty}L_x u_n\right)$$

It is clear that:

$$u_1 = -J^\alpha A_0 + J^\alpha L_{xxx}u_0 - J^\alpha L_{xx}u_0 + J^\alpha L_x u_0,$$

$$u_2 = -J^\alpha A_1 + J^\alpha L_{xxx}u_1 - J^\alpha L_{xx}u_1 + J^\alpha L_x$$

$\vdots$

$$u_{n+1} = -J^\alpha A_n + J^\alpha L_{xxx}u_n - J^\alpha L_{xx} + J^\alpha L_x u_n.$$

By substituting the value of $u_0, u_1, \ldots$

From Eq. (20), we get the solution of the IVP

$u(x,t) = u_0 + u_1 + u_2 + \cdots + u_n + \cdots$

$u_0 = u(x,0) = f(x) = \cos x$ $u_1 = -J^\alpha A_0 + J^\alpha(L_{xxx}u_0) - J^\alpha(L_{xx}u_0) + J^\alpha(L_x u_0)$

$$= \frac{f(x)f'(x) - f'''(x) + f''(x) - f'(x)}{\Gamma(\alpha+1)} t^\alpha$$

$$\therefore u_1 = f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

Where

$$f_1(x) = f(x)f'(x) - f'''(x) + f''(x) - f'(x)$$

$$= (-\sin x - 1)\cos x - 2\sin x$$

$$u_2 = -J^\alpha(A_1) + J^\alpha(L_{xxx}u_1) - J^\alpha(L_{xx}u_1) + J^\alpha(L_x u_1)$$

$$= \frac{f_2(x)}{\Gamma(2\alpha+1)} t^{2\alpha}$$

Where

$$f_2(x) = -[f(x) + f_1'(x) + f_1(x)f'(x) - f_1'''(x) + f_1''(x) - f_1'(x)]$$

$$= \cos^3 x + 3\cos^2 x + ((-\sin x - 1)\sin x - 2\sin x - 1)\cos x - 5\sin^2 x - 2\sin x,$$

similarly

$$u_3 = \frac{f_3(x)}{\Gamma(3\alpha+1)} t^{3\alpha}$$

$\vdots$

$$u_n = \frac{f_n(x)}{\Gamma(n\alpha+1)} t^{n\alpha}$$

$$u(x,t) = f(x) + \frac{t^\alpha}{\Gamma(\alpha+1)} f_1(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} f_2(x) + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} f_n(x) + \cdots$$

$= \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} f_n(x)$ Where $f_0(x)$ is an initial condition.

In the following example, we try to find the solve of another nonlinear fractional equation

Example 3.

Consider the following of nonlinear fractional equation:-

$$D_t^\alpha u + D_x^2 u - D_x u + u^2 = 0$$

$$0 < x \le 1, 0 \le t \le 1, \quad 0 < \alpha \le 1$$

With initial condition

$$u(x,0) = \varphi = f(x) = x^2,$$

$$(x,t) \in \Omega \times [0,T]$$

Note that here $\Omega = (0,1)$

The standard form of the fractional equation an operator form is

$$D_t^\alpha u = -[u(x,t)]^2 - L_{xx}u(x,t) + L_x u(x,t)$$

Where $L_{xx} = \frac{\partial^2}{\partial x^2}, L_x = \frac{\partial}{\partial x}$ and the fractional differential operator D_t^α defined in (2) respectively.

J^α is the inverse of D_t^α

Now applying J^α to the both side of our Eq. we obtain

$$u(x,t) = \Phi - J^\alpha(Nu) - J^\alpha(L_{xx}u(x,t)) + J^\alpha(L_x u(x,t))$$

Where $Nu = u^2$, according to the decomposition method, we assume series solution for the unknown function $u(x,t)$ in the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

In order to solve our problem, we must generalize these Adomian polynomials as follows:

$$An = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}$$

$$, \quad n = 0, 1, \ldots.$$

$$A_0 = u_0 .u_0 = u_0^2$$

$$A_1 = \frac{d}{d\lambda}[(u_0 + \lambda u_1).(u_0 + \lambda u_1)]_{\lambda=0}$$

$$= u_0 u_1 + u_1 u_0 = 2u_0 u_1$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[(u_0 + \lambda u_1 + \lambda^2 u_2)(u_0 + \lambda u_1 + \lambda^2 u_2)]$$

$$= 2u_0 u_2 + u_1^2$$

$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}[(u_0 + \lambda u_1 + \lambda^2 u_2 + \lambda^3 u_3)(u_0 + \lambda u_1 + \lambda^2 u_2 + \lambda^3 u_3)]_{\lambda=0}$$

$$= 2u_0 u_3 + 2u_1 u_2$$

$\vdots$

And so on

From (7), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = \varphi - J^{\alpha}(\sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n)) - J^{\alpha}(\sum_{n=0}^{\infty}(L_{xx}u_n)) + J^{\alpha}(\sum_{n=0}^{\infty}(L_x u_n))$$

From the above equation, we observe that

$$u_0 = u(x,0) = f(x)$$

$$u_1 = -J^{\alpha}(A_0) - J^{\alpha}(L_{xx}u_0) + J^{\alpha}(L_x u_0)$$

$$u_2 = -J^{\alpha}(A_1) - J^{\alpha}(L_{xx}u_1) + J^{\alpha}(L_x u_1)$$

$\vdots$

$$u_{n+1} = -J^{\alpha}(A_n) - J^{\alpha}(L_{xx}u_n) + J^{\alpha}(L_x u_n)$$

By substituting the values of $u_0, u_1, \dots,$

we get solutions of the IVP

$u = u_0 + u_1 + u_2 + \dots + u_n + \dots$

$$u_0 = u(x,0) = f(x) = x^2$$

$$u_1 = -J^{\alpha}(A_0) - J^{\alpha}(L_{xx}u_0) + J^{\alpha}(L_x(u_0))$$

$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)} f_1(x)$$

Where

$$f_1(x) = (f(x))^2 + f''(x) - f'(x) = x^4 - 2x + 2$$

$$\therefore u_1 = x^4 - 2x + 2\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]$$

$$u_2 = -J^{\alpha}(A_1) - J^{\alpha}(L_{xx}u_1) + J^{\alpha}(L_x u_1)$$

$$= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} f_2(x)$$

Where $f_2(x) = -[2f(x)f_1(x) + f_1''(x) - f_1'(x)]$

$$= -2x^6 + 8x^2 + 2$$

$$u_3 = -J^{\alpha}(A_2) - J^{\alpha}(L_{xx}u_2) + J^{\alpha}(L_x u_2)$$

$$= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} f_3(x)$$

Where

$$f_3(x) = -2f(x)f_2(x) - 2f_1^2(x) - f_2''(x) + f_2'(x)$$

$$= 3x^8 - 8x^5 + 40x^4 - 8x^2 + 24x - 20$$

Similarly

$$u_4 = \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} f_4(x)$$

$\vdots$

$$u_n = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} f_n(x)$$

The solution of the considered IVP is given by

$$u(x,t) = f(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)} f_1(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} f_2(x) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} f_n(x)$$

Where $f_0(x) = u(x,0)$, is initial condition.

Also, in the next example, we solve a system of nonlinear equations with fractional orders

Example 4.

Consider the system of initial value problem (IVP) of fractional equations

$D_t^\alpha u = u\, D_x\, u + v\, D_y\, u$

$D_t^\alpha v = u\, D_x v + v\, D_y\, v$

Where$0 < \alpha \leq 1$and$(x,t) \in \Omega_x\ (0,\mathrm{T}]$, and with the initial condition.

$u(x,y,0) = f(x,y)$

$v(x,y,0) = g(x,y) \qquad , x,y \in \Omega$

Not that$\Omega = (0,1)$

The above system can be written in the equivalent form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u\,\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} = u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}$$

$$N_1(u,v) = u\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}$$

$$N_2(u,v) = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

$Lu = N_1(u,v)$

$Lv = N_2(u,v)$

Applying $L^{-1}(.) = J^\alpha$ to both sides ofEq.(28) yields.

$u(x,y,t) = \phi + J^\alpha N_1(u,v)$

$v(x,y,t) = \phi + J^\alpha N_2(u,v)$

Where the nonlinear operator $N_1(u,v)$ and $N_2(u,v)$are them written in the decomposition form

$$N_1(u,v) = \sum_{n=0}^{\infty} A_n\,(u_0,u_1,\dots,u_n)$$

$$N_2(u,v) = \sum_{n=0}^{\infty} B_n\,(v_0,v_1,\dots,v_n)$$

Where A_n and $B_n$are the Adomian polynomials of the following form

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_1(u,v)\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}$$

$$B_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_2(u,v)\left(\sum_{i=0}^{\infty}\lambda^i v_i\right)\right]_{\lambda=0}$$

$$, n = 0,1,\dots$$

generalize these Adomian polynomials in follows.

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\frac{\partial}{\partial x}\sum_{i=0}^{\infty}\lambda^i u_i\right) + \left(\sum_{i=0}^{\infty}\lambda^i v_i\right)\left(\frac{\partial}{\partial y}\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} ,$$

$$n = 0,1,\dots$$

$$A_0 = u_0\frac{\partial u_0}{\partial x} + v_0\,\frac{\partial u_0}{\partial y}$$

$$\therefore A_1 = u_0\frac{\partial u_1}{\partial x} + v_0\,\frac{\partial u_1}{\partial y} + u_1\frac{\partial u_0}{\partial x} + v_1\,\frac{\partial u_0}{\partial y}$$

$$\therefore A_2 = u_0\frac{\partial u_2}{\partial x} + v_0\frac{\partial u_2}{\partial y} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} + u_2\frac{\partial u_0}{\partial x} + v_2\,\frac{\partial u_0}{\partial y}$$

Similarly

$$\therefore A_3 = u_0\frac{\partial u_3}{\partial x} + v_0\frac{\partial u_3}{\partial y} + u_1\frac{\partial u_2}{\partial x} + v_1\frac{\partial u_2}{\partial y} + u_2\frac{\partial u_1}{\partial x} + v_2\frac{\partial u_1}{\partial y} + u_3\frac{\partial u_0}{\partial x} + v_3\frac{\partial u_0}{\partial y}$$

$\vdots$

And so on

Now, to calculated our problem we must generalize these Adomian polynomials in follows

$$B_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\frac{\partial}{\partial x}\sum_{i=0}^{\infty}\lambda^i v_i\right) + \left(\sum_{i=0}^{\infty}\lambda^i v_i\right)\left(\frac{\partial}{\partial y}\sum_{i=0}^{\infty}\lambda^i v_i\right)\right]_{\lambda=0},$$

$$n = 0, 1, 2, \ldots$$

$$B_0 = u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y}$$

$$B_1 = u_0\frac{\partial v_1}{\partial x} + v_0\frac{\partial v_1}{\partial y} + u_1\frac{\partial v_0}{\partial x} + v_1\frac{\partial v_0}{\partial y}$$

$$B_2 = u_0\frac{\partial v_2}{\partial x} + v_0\frac{\partial v_2}{\partial y} + u_1\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_1}{\partial y} + u_2\frac{\partial v_0}{\partial x} + v_2\frac{\partial v_0}{\partial y}$$

$$\therefore B_3 = u_0\frac{\partial v_3}{\partial x} + v_0\frac{\partial v_3}{\partial y} + u_1\frac{\partial v_2}{\partial x} + v_1\frac{\partial v_2}{\partial y} + u_2\frac{\partial v_1}{\partial x} + v_2\frac{\partial v_1}{\partial y} + u_3\frac{\partial v_0}{\partial x} + v_3\frac{\partial v_0}{\partial y}$$

And so on

From (7), we obtain

$$\sum_{n=0}^{\infty} u_n(x,y,t) = u(x,y,0) + $$
$$+J^{\alpha}\left(\textstyle\sum_{n=0}^{\infty} A_n(u_0,u_1,\ldots,u_n)\right)$$

$$\sum_{n=0}^{\infty} v_n(x,y,t) = v(x,y,0) + $$
$$+J^{\alpha}\left(\sum_{n=0}^{\infty} B_n(u_0,u_1,\ldots,u_n)\right)$$

The associated decomposition is given by

$u_0 = u(x,y,0), u_{n+1} = J^{\alpha}(N_1(u_n,v_n))$

$v_0 = v(x,y,0), v_{n+1} = J^{\alpha}(N_2(u_n,v_n))$, n=0,1,2,...

Then, According to the above equations we get,

$u_0 = u(x,y,0)$

$v_0 = v(x,y,0)$

$u_1 = J^{\alpha}A_0$

$v_1 = J^{\alpha}B_0$

$u_2 = J^{\alpha}A_1$

$v_2 = J^{\alpha}B_1$

$\vdots$

$u_{n+1} = J^{\alpha}A_n$

$v_{n+1} = J^{\alpha}B_n$

By substituting the values of $u_0, u_1, \ldots$ And $v_0, v_1, \ldots$ we get a solution of the IVP.

$u_0 = u(x,y,0) = f(x,y)$

$v_0 = v(x,y,0) = g(x,y)$

$$u_1 = J^{\alpha}(A_0) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-\theta)^{1-\alpha}}\left[u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y}\right]d\theta$$
$$= f_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

Where

$$f_1(x,y) = -\left[f(x,y)\frac{\partial f(x,y)}{\partial x} + g(x,y)\frac{\partial f(x,y)}{\partial y}\right]$$

$$v_1 = J^{\alpha}B_0 = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-\theta)^{1-\alpha}}\left[u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y}\right]d\theta$$
$$= g_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

Where

$$g_1(x,y) = -\left[f(x,y)\frac{\partial g(x,y)}{\partial x} + g(x,y)\frac{\partial g(x,y)}{\partial y}\right]$$

$$u_2 = J^{\alpha}(A_1) = f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Where

$$f_2(x) = \left[f(x,y)\frac{\partial f_1(x,y)}{\partial x} + g(x,y)\frac{\partial f_1(x,y)}{\partial y} + f_1(x,y)\frac{f(x,y)}{\partial x} + g_1(x,y)\frac{\partial f(x,y)}{\partial y}\right]$$

$$v_2 = J^{\alpha}B_1 = g_2(x,y)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Where

$$g_2(x,y) = f(x,y)\frac{\partial g_1(x,y)}{\partial x} + g(x,y)\frac{\partial g_1(x,y)}{\partial y} + f_1(x,y)\frac{\partial g(x,y)}{\partial x} + g_1(x,y)\frac{\partial g(x,y)}{\partial y}$$

By induction, we have

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots + u_n + \cdots$$

$$= f(x,y) + f_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f_2(x,y)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + f_n(x,y)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots$$

$$v(x,y,t) = \sum_{n=0}^{\infty} v_n = v_0 + v_1 + \cdots + v_n + \cdots$$

$$= g(x,y) + g_1(x,y)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + g_2(x,y)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + g_n(x,y)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots$$

Finally, we apply the Adomian decomposition method to obtain approximate solutions of IVPs for fractional BBM-Burger's equation with $\mathcal{E} = 1$

Example 5.

Consider the initial value problem (IVP) for fractional BBM_Burger's equation of the form

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0$$

Where $0 < \alpha \leq 1$ and with initial condition

$$u(x,0) = \varphi = f(x) = \sin(x), x \in \Omega \times (0,T]$$

Note that here $\Omega = (0,1)$,the standard form of the fractional BBM_Burger's equation in an operator form

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}$$

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \left(L_{xx}u(x,t)\right) - \left(u(x,t)L_x u(x,t)\right)$$

Where $L_{xx} = \frac{\partial^2}{\partial x^2}$, $L_x = \frac{\partial}{\partial x}$

And the fractional differential operator $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ defined in equation (2), respectively we know that J^{α} which is invers of the operator $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$, now applying J^{α} to the both sides of our Eq., we get

$$u(x,t) = \phi + J^{\alpha}(L_{xx}u) - J^{\alpha}(Nu)$$

Where $Nu = u\frac{\partial u}{\partial x}$

In order to solve our problem we must generalize these Adomian polynomials as follows.

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{n}\lambda^i u_i\right)\frac{\partial}{\partial x}\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0} ,$$

$n = 0, 1, 2, \ldots$

$$A_0 = u_0\frac{\partial u_0}{\partial x}$$

$$A_1 = u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}$$

$$A_2 = u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x}$$

$\vdots$

And so on

Thus

$$u(x,t) = \varphi + J^{\alpha}\left(\sum_{n=0}^{\infty}(L_{xx}u_n)\right) - J^{\alpha}\left(\sum_{n=0}^{\infty}(A_n)\right),$$

$$u_0 = \varphi = u(x,0)$$

$$u_1 = J^{\alpha}(L_{xx}u_0) - J^{\alpha}(A_0)$$

$$u_2 = J^{\alpha}(L_{xx}u_1) - J^{\alpha}(A_1)$$

$\vdots$

$$u_{n+1} = J^{\alpha}(L_{xx}u_n) - J^{\alpha}(A_n)$$

Consequently

$$u(x,t) = u_0 + u_1 + u_2 + \cdots + u_n + \cdots$$

$u_0 = u(x,0) = f(x) = \sin x$

$u_1 = J^{\alpha}(L_{xx}u_0) - J^{\alpha}(A_0) = \frac{f_1(x)}{\Gamma(\alpha+1)}t^{\alpha}$

Where $f_1(x) = -f''(x) + f(x)f'(x)$

$= \sin x)(1 + \cos x)$

$u_2 = J^{\alpha}(L_{xx}u_1) - J^{\alpha}(A_1) = \frac{f_2(x)}{\Gamma(\alpha+1)}t^{\alpha}$

Where

$f_2(x) =$
$f_1''(x) - f(x)f_1'(x) -$
$f_1(x)f'(x)$

$= \sin^3 x + [-1 - 5\cos x - \cos^2 x - (1+\cos x)\cos x]$

$u_3 = J^{\alpha}(L_{xx}u_2) - J^{\alpha}(A_2) = f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$

Similarly

$\therefore u_3 = \frac{f_3(x)}{\Gamma(3\alpha+1)}t^{3\alpha}$

$\vdots$

$u_n = \frac{f_n(x)}{\Gamma(n\alpha+1)}t^{n\alpha}$

The solution of the considered IVP is given by

$u(x,t) = f(x) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}f_1(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}f_2(x) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}f_3(x) + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}f_n(x) + \cdots$

(38)

$= \sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}f_n(x)$,

where $f_0(x)$ is an initial condition. (Comp. 7- 23).

Conclusion

In this paper, we have applied the Adomian decomposition method for solving problems of nonlinear partial equations. We demonstrated that the decomposition procedure is quite efficient to determine the exact solutions. However, the method gives a simple powerful tool for obtaining the solutions without a need for large size of computations. It is also worth noting that the advantage of this method sometimes displays a fast convergence of the solutions. In addition, the numerical results which obtained by this method indicate a high degree of accuracy, also efficiency of the desired results.

REFERENCES

[1] A.A. Kilbas; H.M. Srivastava and J.J. Trujillo, (2006): Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam-Netherlands.

[2] A.H. Ali and A. S. J. Al-Saif, (2008): Adomian decomposition method for solving some models of nonlinear Partial Differential Equations,Basrah Journal of Scienec (A) vol. 26,1-11.

[3] D.B. Dhaigude; Gunvant, A. Birajdar and V.R. Nikam, (2012): Adomian decomposition method for fractional Benjamin-Bona-Mahony-Burger's equation, int. of appl. math, and mech 8 (12): 42-51.

[4] Hosseini, M.M. and Jafari, M.A.(2009): Note on the use of Adomian decomposition method for high-order and system of nonlinear differential equations, communication in nonlinear science and numerical simulation, 14, 1952–1957.

[5] Mahmoud, M. El-Borai, (1980): On the initial value problem for partial differential equations with operator coefficients, Int. J, of Math. And Mathematical Sciences, 3, pp. 103-111.

[6] Mahmoud M. El-Borai and Khairia El- Said El-Nadi, (2011): On some fractional parabolic equations driven by fractional Gaussian noise, Special issue Science and Mathematics With Applications, Int.J.of Research and reviewers in Applied Sciences, 6(3), 236-241.

[7] Mahmoud M. El-Borai, Afaf A. Zaghrout and Amal L. Elshaer, (2011): Exact solutions for nonlinear partial differential equations by using cosine- function method, International Journal of Research and Reviews in Applied Sciences, Vol.9 Issue3 December ,12-17.

[8] Mahmoud M. El-Borai, Afaf A. Zaghrout and Amal L. Elshaer, (2011): Exact solutions for nonlinear partial differential equations by using the extended multiple Riccati equations expansion method, International Journal of Research and Reviews in Applied Sciences, Vol.9 Issue3 December, 17- 25.

[9] Mahmoud M.El-Borai , Mohammed Abdellah and MahaKojok, (2009): Toepletz matrix method and nonlinear integral equations of Hammerstein type, Journal of Computational and Applied Mathematics, 223, 765-776.

[10] Mahmoud M. El-Borai, (2008): Exact solutions for some nonlinear fractional parabolic partial differential equations, Journal of Applied Mathematics and Computation, 206, 141- 15.

[11] Mahmoud M. El-Borai, Wagdy G. Elsayed and Ragab M. Al-Masroub,(2015): Exact solutions for some nonlinear fractional parabolic equations, Inter. J. Adv. Eng. Research, (IJAER), Vol. 10, Issue III, September, 106- 122.

[12] Muhammed, Kurulay, (2010): The approximate and exact solutions of the space-and time fractional Burger's equation, (IJRRAS) 3 (3).

[13] Khairia El-Said El-Nadi, Wagdy G. El-Sayed and Ahmed KhdherQassem, (2015): On some dynamical systems of controlling tumor growth, International Journal of Applied Science and Mathematics, (IJASM), Vol. 2, Issue 5, 146- 151.

[14]Khairia El-Said El-Nadi, Wagdy G. El-Sayed and Ahmed KhdherQassem, (2015): Mathematical model of brain cancer, International Research Journal of Engineering and Technology, (IRJET), Vol. 2, Issue 5, (August), 590 – 594.

[15] S. Saha, Ray, (2014): New approach for general convergence of the Adomian decomposition method, world applied sciences journal 32 (11): 2264-2268, (ISSN) 1818-4952c, idosi publications.

[16] V. Parthiban and K. Balachandran, (2013): Solutions of system of fractional partial differential equations,applications and applied mathematics: an international journal (aam).pp. 289 – 304.

[17] W.G.El-Sayed and J. Banac,(1992): Solvability of Hammerstein integral equation in the class of functions of locally bounded variation, Boll. Un. Itly (7), 5-B, 893-904.

[18] W.G.El-Sayed and J. Banach, (1992): Measures of noncompactness and solvability of anintegral equation in the class of functions of locally bounded variation, J.Math. Anal. Appl.167, 133-151.

[19] W.G. El-Sayed, (1996): A note on a fixed point property for metric projections, Tamk. J. Math., Tamk. Univ.,China, Vol. 27, No. 1, Spring .

[20] WagdyG. El-Sayed and E.M.El-Abd, (2012): Monotonic solutions for a nonlinear functional integral equations of convolution type, J. Fixed Point Theory and App. (JP) Vol.7, No.2, (101-111).

[21] WagdyG.El-Sayed, Mahmoud M.El-Borai, Eman Hamd Allah, and Alaa A.El-Shorbagy, (2013): On some partial differential equations with operator coefficients and non-local conditions, life Science J., 10 (4), (3333-3336).

[22] WagdyG.El-Sayed, Mahmoud M. l-Borai and AmanyM.Moter, (2015): Continuous Solutions of a Quadratic Integral Equation, Inter. J.Life Science and Math. (IJLSM),Vol.2(5)-4, 21-30.

[23] Yihong Wang, Zhengang Zhao, Changpin Li, YangQuan Chen, (2009): Adomian's method applied to navier-stokes equation with a fractional order, international design engineering technical conferences & computers and information in engineering conference. 86691.