Unreliable Batch Arrival Retrial Queue with Negative Arrivals, Multi-Types Of Heterogeneous Service, Feedback and Randomized J Vacations

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Abstract: *A batch arrival retrial queue with two types of arrival - positive and negative is considered. Positive customers arrive in batches according to Poisson process. Server provides M types of service. If the server is idle upon the arrival of a batch, one of the customers in the batch receives any one of the types immediately and others join the orbit. Otherwise, all the customers in the batch join the orbit. After completion of service, the unsatisfied customers join the orbit as feedback customers. Whenever the orbit becomes empty the server takes a vacation of random length. At each vacation completion epoch, if the system is empty, the server leaves for another vacation of same length or remains idle in the system. This pattern continues until the server finds at least one customer in the orbit or the number of vacations reaches J. At the end of J^{th} vacation, even if the orbit is empty the server remains in the system for new arrival. The arrival of negative customer brings the server down and makes the interrupted customer to leave the system. Using supplementary variable technique, various performance measures are derived. Stochastic decomposition property is established and particular cases are deduced.*

Keywords: Retrial queue, negative customers, feedback, randomized J vacation.

1. INTRODUCTION

Queueing systems with vacation have been found to be useful in modeling the production systems, communication systems and manufacturing systems. Chang and Ke[2] introduced the concept of modified vacation policy in which on finding the empty orbit, the server takes at most J vacations repeatedly until at least one customer is in the orbit. Jain and Charu Bhargava [8] suggested an M/G/1 retrial queueing system with modified vacation and K phase repair and obtained the probability generating function of the steady state queue size at random epoch. Chen et al.[3] investigated a retrial queue with modified vacation and server breakdown. Rajadurai et al.[10] analysed the steady state batch arrival retrial queueing system with two types of service under modified vacation policy.

In recent years there has been a rapid increase in the literature on retrial queueing system with negative customers. Queue with negative arrivals called G-queue was introduced by Gelenbe [4] with a view to modeling neural networks. The arrival of negative customer brings the server down and makes the interrupted customer to leave the system. For a comprehensive analysis of queueing systems with negative arrivals, reader may refer to Gelenbe[5-7] and Artalejo[1]. Liu et al. [9] analysed an M/G/1 retrial G-queue with preemptive resume and feedback under N-policy vacation. Wu and Yin [11] considered an unreliable M/G/1 retrial G-queue with non exhaustive random vacation and derived steady state solution for both queueing measures and reliability quantities.

In this paper, batch arrival retrial queue with negative customers, multi-types of service and feedback is analyzed under randomized J vacation policy. Under this policy, at the end of each vacation if no customers are waiting for service, the server leaves for another vacation with certain probability or remains in the system with complementary probability.

2. MODEL DESCRIPTION

Single server queueing system with two types of arrivals - positive and negative is considered. Positive customers arrive in batches according to Poisson process with rate λ^+. At every arrival epoch, a batch of k customers arrive with probability C_k. The generating function of the sequence $\{C_k\}$ is C(z) with first two moments m_1 and m_2. The server provides M types of service. Customers opt the i^{th} type of service with probability p_i $(1 \leq i \leq M)$. There is no waiting space in front of the server and therefore if the arriving batch of positive customers finds the server idle, then one of the customers receives service and the others join the orbit. After receiving service, the customer may again join the orbit as a feedback customer with probability δ or depart the system with its complementary probability $\bar{\delta}$ (=1- δ).

At a service completion epoch if the orbit is empty, the server leaves for a vacation of random length V. At a vacation completion epoch, if the orbit is still empty the server either remains idle in the system with probability q or takes another vacation with probability $\bar{q}$. This pattern continues until the server finds at least one customer in the orbit or number of vacations reaches J. At the end of J^{th} vacation, even if the orbit is empty the server remains in the system.

Negative customers arrive independently according to Poisson process with rate λ^-. The arrival of negative customer removes the positive customer in service from the system and causes the server breakdown. The repair of the failed server commences immediately.

Distribution function, density function, Laplace Stieltje's transform and the first two moments of retrial time, service time, repair time and vacation time which are generally distributed are given below.

Time	Distri-bution function	Density function	Laplace **Stieltje's** Transform	First two moments	Hazard rate function
Retrial	$A(x)$	$a(x)$	$A^*(s)$	-	$\eta(x)$
Service	$B_i(x)$	$b_i(x)$	$B_i^*(s)$	$\mu_i^{(1)}$, $\mu_i^{(2)}$	$\mu_i(x)$
Repair	$R_i(x)$	$r_i(x)$	$R_i^*(s)$	$\beta_i^{(1)}$, $\beta_i^{(2)}$	$\beta_i(x)$
Vacation	$V(x)$	$v(x)$	$V^*(s)$	v_1, v_2	$\gamma(x)$

The stochastic behavior of the retrial queueing system can be described by the Markov process $\{X(t), t \geq 0\} = \{C(t), N(t), \xi(t), t \geq 0\}$ where $C(t)$ denotes the server state 0, i, M+i, 2M+j according as the server being idle, busy in i^{th} type service, under repair which is failed during i^{th} type service or on j^{th} vacation. $N(t)$ corresponds to the number of customers in the orbit. If $C(t) = 0$, then $\xi(t)$ represents the elapsed retrial time. If $C(t) = i$, then $\xi(t)$ represents the elapsed type i service time. If $C(t) = M+i$, then $\xi(t)$ represents the elapsed repair time of the server failed during i^{th} type service, $(1 \leq i \leq M)$. If $C(t)=2M+j$, then $\xi(t)$ represents the elapsed j^{th} vacation time$(1 \leq j \leq J)$.

3. STEADY STATE ANALYSIS

For the process $\{X(t), t \geq 0\}$, define the probability densities

$I_0(t) = P\{C(t) = 0, N(t) = n\}$

$I_n(x,t) = P\{C(t) = 0, N(t) = n, x \leq \xi(t) < x+dx\}, x \geq 0, n \geq 1$

$P_n^{(i)}(x,t)dx = P\{C(t)=i, N(t)=n, x \leq \xi(t) < x+dx\}, x \geq 0, n \geq 0,$
$i = 1,2,\ldots,M$

$R_n^{(i)}(x,t)dx = P\{C(t)=i+M, N(t)=n, x \leq \xi(t) < x+dx\}, x \geq 0, n \geq 0,$
$i = 1,2,\ldots,M$

$V_n^{(j)}(x,t)dx = P\{C(t)=2M+j, N(t)=n, x \leq \xi(t) < x+dx\}, x \geq 0, n \geq 0,$
$j = 1,2,\ldots,J.$

The system of equilibrium equations that governs the model under consideration are

$$\lambda^+ I_0 = \int_0^\infty V_0^{(J)}(x)\gamma(x)dx + q\sum_{j=1}^{J-1}\int_0^\infty V_0^{(j)}(x)\gamma(x)dx \tag{1}$$

$$\frac{d}{dx} I_n(x) = -(\lambda^+ + \eta(x)) I_n(x), n \geq 1 \tag{2}$$

$$\frac{d}{dx} P_n^{(i)}(x) = -(\lambda^+ + \lambda^- + \mu_i(x)) P_n^{(i)}(x) + (1-\delta_{n0})\lambda^+ \sum_{k=1}^{n} C_k P_{n-k}^{(i)}(x), \quad n \geq 0, i = 1,2,\ldots,M \tag{3}$$

$$\frac{d}{dx} R_n^{(i)}(x) = -(\lambda^+ + \beta_i(x)) R_n^{(i)}(x) + (1-\delta_{n0})\lambda^+ \sum_{k=1}^{n} C_k R_{n-k}^{(i)}(x), \quad n \geq 0, i = 1,2,\ldots,M \tag{4}$$

$$\frac{d}{dx} V_n^{(j)}(x) = -(\lambda^+ + \gamma(x)) V_n^{(j)}(x) + (1-\delta_{n0})\lambda^+ \sum_{k=1}^{n} C_k V_{n-k}^{(j)}(x), \quad n \geq 0, j = 1,2,\ldots,J \tag{5}$$

with boundary conditions

$$I_n(0) = \sum_{i=1}^{M}\left[\delta\int_0^\infty P_{n-1}^{(i)}(x)\mu_i(x)dx + \bar{\delta}\int_0^\infty P_n^{(i)}(x)\mu_i(x)dx + \int_0^\infty R_n^{(i)}(x)\beta_i(x)dx\right] + \sum_{j=1}^{J}\int_0^\infty V_n^{(j)}(x)\gamma(x)dx, n \geq 1 \tag{6}$$

$$P_0^{(i)}(0) = p_i\left[\lambda^+ c_1 I_0 + \int_0^\infty I_1(x)\eta(x)dx\right] \tag{7}$$

$$P_n^{(i)}(0) = p_i\left[\begin{array}{l}\lambda^+ c_{n+1} I_0 + \int_0^\infty I_{n+1}(x)\eta(x)dx + \\ \lambda^+ \sum_{k=1}^{n} c_k \int_0^\infty I_{n-k+1}(x)dx\end{array}\right], n \geq 1, i = 1,2,\ldots,M \tag{8}$$

$$R_n^{(i)}(0) = \lambda^- \int_0^\infty P_n^{(i)}(x)dx, n \geq 0, i = 1,2,\ldots,M \tag{9}$$

$$V_n^{(j)}(0) = \begin{cases}\bar{q}\int_0^\infty V_n^{(j-1)}(x)\gamma(x)dx, & n = 0, j = 2,3,\ldots,J \\ 0 & , n \neq 0, j = 2,3,\ldots,J\end{cases} \tag{10}$$

$$V_n^{(1)}(0) = \begin{cases}\sum_{i=1}^{M}\left(\bar{\delta}\int_0^\infty P_n^{(i)}(x)\mu_i(x)dx + \int_0^\infty R_n^{(i)}(x)\beta_i(x)dx\right), & n = 0 \\ 0 & , n \neq 0\end{cases} \tag{11}$$

Define the following probability generating functions for $|z| \leq 1$

$I(x,z) = \sum_{n=1}^{\infty} I_n(x)z^n$, $P^{(i)}(x,z) = \sum_{n=0}^{\infty} P_n^{(i)}(x)z^n$,

$R^{(i)}(x,z) = \sum_{n=0}^{\infty} R_n^{(i)}(x)z^n$, $i=1,2,\ldots,M$ and

$V^{(j)}(x,z) = \sum_{n=0}^{\infty} V_n^{(j)}(x)z^n, j = 1,2,\ldots,J$

3.1. Theorem:

The joint steady state distributions of the server state and orbit size are given by

$$I(z) = I_0(1 - A^*(\lambda^+))\left[\begin{array}{l}C(z)\sum_{i=1}^{M} p_i[(\delta(z-1)+1)g(z)B_i^*(g(z)) + \\ \lambda^-(1 - B_i^*(g(z))R_i^*(h(z))] \\ + zg(z)[N(V^*(h(z))-1)-1]\end{array}\right] / D(z) \tag{12}$$

$$P^{(i)}(z) = I_0\lambda^+ p_i[1-B_i^*(g(z))][A^*(\lambda^+)(C(z)-1)+ [A^*(\lambda^+)+C(z))(1-A^*(\lambda^+))]N(V^*(h(z))-1)]/D(z), \quad 1\le i\le M \tag{13}$$

$$R^{(i)}(z) = I_0\lambda^- p_i[1-B_i^*(g(z))][1-R_i^*(h(z))][A^*(\lambda^+)(C(z)-1)+ [A^*(\lambda^+)+C(z))(1-A^*(\lambda^+))]N(V*(h(z))-1)]/[(1-C(z))D(z)], \quad 1\le i\le M \tag{14}$$

$$V^{(j)}(z) = \frac{I_0\, N\, [1-V^*(h(z))]}{(1-C(z))}, \quad 1\le j\le J \tag{15}$$

$$I_0 = \frac{\lambda^-[1-m_1+m_1A^*(\lambda^+)]-\lambda^+m_1[1-\sum_{i=1}^{M} p_iB_i^*(\lambda^-)]-\sum_{i=1}^{M} p_i[\lambda^-\delta B_i^*(\lambda^-)+\lambda^+\lambda^-m_1\beta_i^{(1)}(1-B_i^*(\lambda^-))]}{(\lambda^+\lambda^-v_1N+\lambda^-A^*(\lambda^+))[1-\delta\sum_{i=1}^{M} p_iB_i^*(\lambda^-)]} \tag{16}$$

where,

$$D(z) = zg(z)-[A^*(\lambda^+)+C(z))(1-A^*(\lambda^+))]\sum_{i=1}^{M} p_i[(\delta(z-1)+1)g(z)B_i^*(g(z)) +\lambda^-(1-B_i^*(g(z))R_i^*(h(z))]$$

$g(z)=\lambda^+ + \lambda^- - \lambda^+C(z)$, $h(z)= \lambda^+ - \lambda^+C(z)$,

$$N = \frac{1-\left(\bar{q}V^*(\lambda^+)\right)^J}{V^*(\lambda^+)[(\bar{q}V^*(\lambda^+))^{J-1}(\bar{q}V^*(\lambda^+)-1)+q((\bar{q}V^*(\lambda^+))^{J-1}-1)]}$$

Proof:

Multiplying equations (2) to (11) by z^n and summing over all possible values of n, we get the following partial differential equations

$$\left[\frac{\partial}{\partial x}+(\lambda^+ +\eta(x))\right]I(x,z)=0 \tag{17}$$

$$\left[\frac{\partial}{\partial x}+(\lambda^+ +\lambda^- -\lambda^+C(z)+\mu_i(x))\right]P^{(i)}(x,z)=0, i=1,2,\ldots,M \tag{18}$$

$$\left[\frac{\partial}{\partial x}+(\lambda^+ -\lambda^+C(z)+\beta_i(x))\right]R^{(i)}(x,z)=0, i=1,2,\ldots,M \tag{19}$$

$$\left[\frac{\partial}{\partial x}+(\lambda^+ -\lambda^+C(z)+\gamma(x))\right]V^{(j)}(x,z)=0, \quad 1\le j\le J \tag{20}$$

$$I(0,z) = \sum_{j=1}^{J}\int_0^\infty V^{(j)}(x,z)\gamma(x)dx + (\delta(z-1)+1)\sum_{i=1}^{M}\left[\int_0^\infty P^{(i)}(x,z)\mu_i(x)dx+\int_0^\infty R^{(i)}(x,z)\beta_i(x)dx\right] -\lambda^+I_0-\sum_{j=1}^{J}V_0^{(j)}(0) \tag{21}$$

$$P^{(i)}(0,z) = \frac{p_i}{z}\left[\lambda^+C(z)I_0+\int_0^\infty I(x,z)\eta(x)dx+\lambda^+C(z)\int_0^\infty I(x,z)dx\right], i=1,2,\ldots,M \tag{22}$$

$$R^{(i)}(0,z) = \lambda^-\int_0^\infty P^{(i)}(x,z)dx, \quad i=1,2,\ldots,M \tag{23}$$

Solving equation (5) at n=0, we have

$$V_0^{(j)}(x) = V_0^{(j)}(0)e^{-\lambda^+x}[1-V(x)], j=1,2,\ldots,J \tag{24}$$

Multiplying equation (24) by $\gamma(x)$ and integrating with respect to x from 0 to ∞, we have

$$\int_0^\infty V_0^{(j)}(x)\gamma(x)dx = \int_0^\infty V_0^{(j)}(0)e^{-\lambda^+x}(1-V(x))\gamma(x)dx = V_0^{(j)}(0)V^*(\lambda^+), j=1,2,\ldots,J \tag{25}$$

Equation (10) gives

$$V_0^{(j)}(0)=\bar{q}\,V_0^{(j-1)}(0)V^*(\lambda^+), j=2,3,\ldots,J \tag{26}$$

From equations (10) and (11) it is clear that

$V^{(j)}(0,z) = V^{(j)}_0(0)$

Applying equations (26) repeatedly for j=J,J-1,... we get

$$V^{(j)}(0,z) = \frac{V_0^{(J)}(0)}{\left[\bar{p}V^*(\lambda^+)\right]^{J-1}}, j=1,2,\ldots,J-1 \tag{27}$$

Substituting equation (26) and (27) in equation (1) and after some algebraic manipulations, we get

$$V_0^{(J)}(0)=\frac{\lambda^+I_0}{V^*(\lambda^+)\left[1+\frac{q(1-(\bar{q}V^*(\lambda^+))^{J-1})}{(\bar{q}V^*(\lambda^+))^{J-1}(1-(\bar{q}V^*(\lambda^+))}\right]} \tag{28}$$

Solving the partial differential equations(17), (18),(19) and(20), we get respectively

$I(x,z)= I(0,z)\exp[-\lambda^+x][1- A(x)]$ (29)

$P^{(i)}(x,z)=P^{(i)}(0,z)\exp[-(g(z))x][1-B_i(x)], i=1,2,\ldots,M$ (30)

$R^{(i)}(x,z)= R^{(i)}(0,z)\exp[-h(z)x][1-R_i(x)], i=1,2,\ldots,M$ (31)

$V^{(j)}(x,z)=V^{(j)}(0,z)\exp[h(z)x][1-V(x)], j=1,2,\ldots,J$ (32)

Substituting the expression of $P^{(i)}(x,z)$ in equation(23), we obtain

$$R^{(i)}(0,z)=\lambda^-P^{(i)}(0,z)(1-B_i^*(g(z))/g(z), i=1,2,\ldots,M \tag{33}$$

Using equations (29) in (22), we get

$$P^{(i)}(0,z)=\frac{p_i}{z}\left[\lambda^+I_0C(z)+I(0,z)(A^*(\lambda^+)+C(z)(1-A^*(\lambda^+)))\right], i=1,2,\ldots,M \tag{34}$$

Equation (21) yields

$$I(0,z)=\lambda^{+}I_0\begin{bmatrix}C(z)\sum_{i=1}^{M}p_i[(\delta(z-1)+1)g(z)B_i^{*}(g(z))\\+\lambda^{-}(1-B_i^{*}(g(z))R_i^{*}(h(z))]\\+zg(z)[N(V^{*}(h(z))-1)-1]\end{bmatrix}/D(z) \tag{35}$$

Solving equation (34) by using equation (35), we obtain

$$P^{(i)}(0,z)=I_0\lambda^{+}p_i g(z)[A^{*}(\lambda^{+})(C(z)-1)+[A^{*}(\lambda^{+})+C(z))(1-A^{*}(\lambda^{+}))]N(V^{*}(h(z))-1)]/D(z) \tag{36}$$

Using equations (36) in equation (33) and simplifying we get

$$R^{(i)}(0,z)=I_0\lambda^{+}\lambda^{-}p_i[1-B_i^{*}(g(z))][A^{*}(\lambda^{+})(C(z)-1)+[A^{*}(\lambda^{+})+C(z))(1-A^{*}(\lambda^{+}))]N(V^{*}(h(z))-1)]/D(z) \tag{37}$$

Using the expressions of $I(0,z)$, $P^{(i)}(0,z)$, $R^{(i)}(0,z)$ and $V^{(j)}(0,z)$, the probability generating functions $I(x,z)$, $P^{(i)}(x,z)$, $R^{(i)}(x,z)$ and $V^{(j)}(x,z)$ can be expressed in terms of I_0. Integrating the resultant expressions with respect to x from 0 to ∞, we get $I(z)$, $P^{(i)}(z)$, $R^{(i)}(z)$ and $V^{(j)}(z)$ as in equations (12),(13),(14) and (15). The expression of I_0 given in equation (16) can be obtained by using the normalizing condition.

4. PERFORMANCE MEASURES

Probability that the server is idle in the non-empty system is given by

$$I=\lim_{z\to 1}I(z)=\frac{\begin{array}{c}(1-A^{*}(\lambda^{+}))(\lambda^{-}[m_1-1+\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]\\+\lambda^{+}m_1[1+\lambda^{-}v_1N+\sum_{i=1}^{M}p_i[-B_i^{*}(\lambda^{-})+\lambda^{-}\beta_i^{(1)}(1-B_i^{*}(\lambda^{-}))]\end{array}}{(\lambda^{+}\lambda^{-}v_1N+\lambda^{-}A^{*}(\lambda^{+}))[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]} \tag{38}$$

Probability that the server is busy is given by

$$P=\lim_{z\to 1}\sum_{i=1}^{M}P^{(i)}(z)=\frac{\sum_{i=1}^{M}\lambda^{+}m_1p_i(1-B_i^{*}(\lambda^{-}))}{\lambda^{-}[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]} \tag{39}$$

Probability that the server is under repair is given by

$$R=\lim_{z\to 1}\sum_{i=1}^{M}R^{(i)}(z)=\frac{\lambda^{+}m_1\sum_{i=1}^{M}\beta_i^{(1)}p_i(1-B_i^{*}(\lambda^{-}))}{[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]} \tag{40}$$

Probability that the server is on vacation is given by

$$V=\lim_{z\to 1}\sum_{j=1}^{J}V^{(j)}(z)$$

$$=\frac{\begin{array}{c}\lambda^{+}v_1N[\lambda^{-}[1-m_1+m_1A^{*}(\lambda^{+})]-\lambda^{+}m_1[1-\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]\\-\sum_{i=1}^{M}p_i[\lambda^{-}\delta B_i^{*}(\lambda^{-})+\lambda^{+}\lambda^{-}m_1\beta_i^{(1)}(1-B_i^{*}(\lambda^{-}))]\end{array}}{(\lambda^{+}\lambda^{-}v_1N+\lambda^{-}A^{*}(\lambda^{+}))[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]} \tag{41}$$

Probability generating function of the number of customers in the orbit is given by

$$P_q(z)=I_0+I(z)+\sum_{i=1}^{M}[P^{(i)}(z)+R^{(i)}(z)]+\sum_{j=1}^{J}V^{(j)}(z)$$

$$=I_0(z-1)g(z)[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(g(z))][(A^{*}(\lambda^{+})+C(z)(1-A^{*}(\lambda^{+}))[N(1-V^{*}(h(z))]+A^{*}(\lambda^{+})(1-C(z))]/[(1-C(z))D(z)] \tag{42}$$

Mean number of customers in the orbit L_q is given by

$$L_q=\lim_{z\to 1}\frac{d}{dz}P_q(z)$$

Let Nr(z) and Dr(z) be the numerator and denominator of $P_q(z)$. Since $Nr(1)=Dr(1)=Nr'(1)=Dr'(1)=0$, applying L'Hospital rule we get

$$L_q=P_q'(1)=\frac{[Dr''(1)Nr'''(1)-Nr''(1)Dr'''(1)]}{3Dr''(1)^2} \tag{43}$$

where

$$Nr''(1)=-I_0 2\lambda^{-}m_1[A^{*}(\lambda^{+})+\lambda^{+}v_1N][1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]$$

$$Nr'''(1)=I_0[6((-m_1A^{*}(\lambda^{+})-\lambda^{+}m_1v_1N)(-\delta\lambda^{+}\lambda^{-}m_1\sum_{i=1}^{M}p_i\mu_i^{(1)}-\lambda^{+}m_1[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})])+3\lambda^{-}[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})](-m_2A^{*}(\lambda^{+})-2\lambda^{+}m_1^2v_1N(1-A^{*}(\lambda^{+}))-N(\lambda^{+}m_2v_1+\lambda^{+2}m_1^2v_2))]$$

$$Dr''(1)=-2m_1\{\lambda^{-}[1-m_1+m_1A^{*}(\lambda^{+}))-\lambda^{+}m_1[1-\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]-\lambda^{-}\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})-\lambda^{+}\lambda^{-}m_1\sum_{i=1}^{M}p_i\beta_i^{(1)})(1-B_i^{*}(\lambda^{-}))\}$$

$$Dr'''(1)=-3m_1[-2\lambda^{+}m_1-\lambda^{+}m_2-\lambda^{-}m_2(1-A^{*}(\lambda^{+}))-\sum_{i=1}^{M}p_i[2\lambda^{+}\lambda^{-}m_1\delta\mu_i^{(1)}-2\lambda^{+}m_1\delta B_i^{*}(\lambda^{-})-\lambda^{+}m_2B_i^{*}(\lambda^{-})+2\lambda^{+2}m_1^2\lambda^{-}\mu_i^{(1)}\beta_i^{(1)}+\lambda^{-}(1-B_i^{*}(\lambda^{-}))[\lambda^{+2}m_1^2\beta_i^{(2)}+\lambda^{+}m_2\beta_i^{(1)}]-2\lambda^{+2}m_1^2\mu_i^{(1)}]-2m_1(1-A^{*}(\lambda^{+}))\sum_{i=1}^{M}p_i[(\lambda^{-}\delta B_i^{*}(\lambda^{-})-\lambda^{+}m_1B_i^{*}(\lambda^{-})+\lambda^{+}\lambda^{-}m_1\beta_i^{(1)}(1-B_i^{*}(\lambda^{-}))]]-3m_2[\lambda^{-}[1-m_1+m_1A^{*}(\lambda^{+}))-\lambda^{+}m_1[1-\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]-\lambda^{-}\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})-\lambda^{+}\lambda^{-}m_1\sum_{i=1}^{M}p_i\beta_i^{(1)})(1-B_i^{*}(\lambda^{-}))]$$

Probability generating function of the number of customers in the system is given by

$$P_s(z)=I_0+I(z)+\sum_{i=1}^{M}[zP^{(i)}(z)+R^{(i)}(z)]+\sum_{j=1}^{J}V^{(j)}(z)$$

$$= I_0(z-1)[\lambda^- + \sum_{i=1}^{M} p_i B_i^*(g(z))(h(z)-\delta g(z))]\,[(A^*(\lambda^+) + C(z)(1-A^*(\lambda^+)))$$
$$[N(1-V^*(h(z))] + A^*(\lambda^+)(1-C(z))]/[(1-C(z))D(z)] \tag{44}$$

Mean number of customers in the orbit L_s is given by

$$L_s = \lim_{z\to 1}\frac{d}{dz}P_s(z)$$

Let $Nr_1(z)$ be the numerator of $P_s(z)$.

$$L_s = P_s'(1) = \frac{\left\lfloor Dr''(1)Nr_1'''(1) - Nr_1''(1)Dr(1)''' \right\rfloor}{3Dr''(1)^2} \tag{45}$$

where

$$Nr_1''(1) = -2\lambda^- m_1[A^*(\lambda^+) + \lambda^+ v_1 N][1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)]I_0$$

$$Nr_1'''(1) = I_0[6((-m_1 A^*(\lambda^+) - \lambda^+ m_1 v_1 N)(-\delta\lambda^+\lambda^- m_1 \sum_{i=1}^{M} p_i \mu_i^{(1)} -$$
$$\lambda^+ m_1[(1-\delta)\sum_{i=1}^{M} p_i B_i^*(\lambda^-)])) + 3(\lambda^-[1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)](-m_2 A^*(\lambda^+)$$
$$-2\lambda^+ m_1^2 v_1 N(1-A^*(\lambda^+)) - N(\lambda^+ m_2 v_1 + \lambda^{+2} m_1^2 v_2))]$$

5. RELIABILITY INDICES

Let A(t) be the pointwise availability of the server at time t, that is the probability that the server is idle or busy.
The steady state availability of the server will be $A = \lim_{t\to\infty} A(t).$

The availability of the server is given by

$$A = 1 - \frac{\begin{array}{l}\lambda^+ m_1 \sum_{i=1}^{M} \beta_i^{(1)} p_i (1-B_i^*(\lambda^-))(\lambda^+\lambda^- v_1 N + \lambda^- A^*(\lambda^+)) \\ +\lambda^+ v_1 N[\lambda^-[1-m_1+m_1A^*(\lambda^+)] - \lambda^+ m_1[1-\sum_{i=1}^{M} p_i B_i^*(\lambda^-)] \\ -\sum_{i=1}^{M} p_i[\lambda^-\delta B_i^*(\lambda^-) + \lambda^+\lambda^- m_1 \beta_i^{(1)}(1-B_i^*(\lambda^-))]\end{array}}{[1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)]} \tag{46}$$

Steady state failure frequency of the server is

$$F = \lambda^- P = \frac{\lambda^+ m_1 \sum_{i=1}^{M} p_i(1-B_i^*(\lambda^-))}{[1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)]} \tag{47}$$

6. STOCHASTIC DECOMPOSTION

6.1. Theorem: The number of customers in the system (L_s) can be expressed as the sum of two independent random variables, one of which is the mean number of customers (L) in the unreliable batch arrival G-queue with multi-type of heterogeneous service and feedback and the other is the mean number of customers in the orbit (L_I)given that the server is idle or on vacation.

Proof: The probability generating function $\pi(z)$ of the number of customers in the classical queueing system with negative customers, server breakdown and feedback is

$$\pi(z) = \frac{I_0(z-1)[\lambda^- + \sum_{i=1}^{M} p_i B_i^*(g(z))(h(z)-\delta g(z))](1-C(z))}{(1-C(z))\begin{bmatrix} zg(z) - \sum_{i=1}^{M} p_i[(\delta(z-1)+1)g(z)B_i^*(g(z)) \\ +\lambda^-(1-B_i^*(g(z))R_i^*(h(z))] \end{bmatrix}} \tag{48}$$

$$\text{where, } I_0 = \frac{\begin{array}{c}\lambda^- - \lambda^+ m_1[1-\sum_{i=1}^{M} p_i B_i^*(\lambda^-)] - \\ \sum_{i=1}^{M} p_i[\lambda^-\delta B_i^*(\lambda^-) + \lambda^+\lambda^- m_1 \beta_i^{(1)}(1-B_i^*(\lambda^-))]\end{array}}{\lambda^-[1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)]}$$

The probability generating function $\chi(z)$ of the number of customers in the orbit given that the server is idle or on vacation is

$$\chi(z) = \frac{I_0 + I(z) + \sum_{j=1}^{J} V^{(j)}(z)}{I_0 + I + V} \tag{49}$$

$$= \frac{\begin{array}{l}\left(A^*(\lambda^+)(C(z)-1) + M(V^*(h(z))-1)[C(z)+A^*(\lambda^+)(1-C(z))]\right) \\ \left(\sum_{i=1}^{M} p_i[(\delta(z-1)+1)g(z)B_i^*(g(z)) + \lambda^-(1-B_i^*(g(z))R_i^*(h(z))] - zg(z)\right) \\ \left(\begin{array}{l}\lambda^-[1-m_1+m_1A^*\lambda^+)] - \lambda^+ m_1[1-\sum_{i=1}^{M} p_i B_i^*(\lambda^-)] - \\ \lambda^-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-) - \lambda^+\lambda^- m_1 \sum_{i=1}^{M} p_i \beta_i^{(1)})(1-B_i^*(\lambda^-))\end{array}\right)\end{array}}{\begin{array}{l}(1-C(z))D(z)[A^*(\lambda^+) + \lambda^+ v_1 N][\lambda^- - \lambda^+ m_1[1-\sum_{i=1}^{M} p_i B_i^*(\lambda^-)] - \\ \lambda^-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-) - \lambda^+\lambda^- m_1 \sum_{i=1}^{M} p_i \beta_i^{(1)})(1-B_i^*(\lambda^-))]\end{array}}$$

From equations (44),(48) and (49), we see that
$P_s(z) = \pi(z)\,\chi(z)$. Hence, $L_s = L + L_I$.

7. CONCLUSIONS

In this paper, a single server batch arrival retrial queue with negative customers, multi-types of heterogeneous service with feedback and randomized J vacation is analyzed. Using supplementary variable technique, analytical expressions for various performance measures are derived. Stochastic decomposition property is also verified. Numerical results are carried out to study the effect of some key parameters on the performance measures of the model.

REFERENCES

[1] J.R. Artalejo, G – newtorks : a versatile approach for work removal in queueing networks. European Journal of Operational Research,2000, 126 , 233-249.
[2] F.M. Chang and J.C. Ke,On a batch retrial model with J vacations, Journal of Computational and Applied

Mathematics, 2009, 232, 402-414.
[3]P. Chen, P., Y. Zhu, and Y. Zhang, A retrial queue with modified vacations and server breakdowns,2010, IEEE, 978 – 1 4244 - 5540 -9, 26-30.
[4] E. Gelenbe, Random neural networks with negative and positive signals and product form solution,1989,Neural Computation1, 502-510.
[5] E. Gelenbe. Product - form queueing networks with negative and positive customers. Journal of applied probability ,1991,28 , 656-663.
[6] E. Gelenbe, G-networks: A unifying model for neural and queueing networks. Annals of Operations Research,1994, 48, 433-461.
[7] E. Gelenbe,. The first decade of G-networks. European Journal of Operational Research,2000, 126 , 231-232.
[8] M. Jain and Charu Bhargava, Unreliable server M / G / 1 queueing system with bernoulli feedback, repeated attempts, modified vacation, phase repair and discouragement. Journal of King Abdulaziz University Engineering Sciences,2009. 20(2), 45-77.
[9] Z. Liu, J. Wu. and G. Yang, An M / G / 1 Retrial G – queue with preemptive resume and feedback under N - policy subject to server breakdowns and repairs, Computers and Mathematics with Applications,2009, 58, 1792-1807.
[10] P. Rajadurai, M.C, Saravanarajan and V.M. Chandrasekaran, Analysis of an $M^{[x]}$ / (G_1,G_2) / 1 Retrial queueing system with balking, optional re-service under modified vacation policy and service interruption, Ain Shams Engineering Journal,2014, dx.doi.org/10.1016/j.asej.2014. 02.00.3.
[11] J. Wu and X. Yin, An M / G / 1 retrial G - queue with non – exhaustive random vacations and an unreliable server, Computers and Mathematics with Applications, 2011,62, 2314-2329.

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