ON THE BINARY QUADRATIC DIOPHANTINE EQUATION

$$x^2 - 3xy + y^2 + 18x = 0$$

S.Vidhyalakshmi [1], M.A.Gopalan [2], S.Nandhini [3]

[1,2] *Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu,*

E-mail: vidhyasigc@gmail.com, mayilgopalan@gmail.com,

[3] *M.Phil Scholar, Department of Mathematics, SIGC,Trichy-620002,Tamilnadu*

E-mail: nandhinisampath14@gmail.com

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Abstract: The binary quadratic equation $x^2 - 3xy + y^2 + 18x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

Key Words: Binary quadratic equation, Integral solutions.

MSC subject classification: 11D09.

1. INTRODUCTION:

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety $[1-6]$. In $[7-16]$ the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of an another interesting binary quadratic equation given by $x^2 - 3xy + y^2 + 18x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. METHOD OF ANALYSIS:

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 3xy + y^2 + 18x = 0 \qquad (1)$$

Note that (1) is satisfied by the following non-zero integer pairs

(18,18), (18,36), (-18,-54), (36,72), (-54,-162).

However, we have solutions for (1), which are illustrated below:

Solving (1) for x, we've

$$x = \frac{1}{2}[(3y - 18) \pm \sqrt{5y^{2} - 108y + 324}] \qquad (2)$$

Let $\alpha^2 = 5y^2 - 108y + 324$

which is written as

$$(5y - 54)^2 = 5\alpha^2 + 36^2$$

$$\Rightarrow Y^2 = 5\alpha^2 + 36^2 \qquad (3)$$

Where

$$Y = 5y - 54 \qquad (4)$$

The least positive integer solution of (3) is

$$\alpha_0 = 144 \ , \ Y_0 = 324$$

Now, to find the other solution of (3), consider the Pellian equation

$$Y^2 = 5\alpha^2 + 1 \qquad (5)$$

whose fundamental solution is $(\tilde{\alpha}_0, \tilde{Y}_0) = (4,9)$

The other solutions of (5) can be derived from the relations

$$\tilde{Y}_n = \frac{f_n}{2} \qquad \tilde{\alpha}_n = \frac{g_n}{2\sqrt{15}}$$

where

$$f_n = [(9+4\sqrt{15})^{n+1} + (9-4\sqrt{15})^{n+1}]$$

$$g_n = [(9+4\sqrt{15})^{n+1} + (9-4\sqrt{15})^{n+1}] \quad ,$$

n=0,2,4,........

Applying the lemma of Brahmagupta between $(\alpha_0, Y_0) \& (\tilde{\alpha}_n, \tilde{Y}_n)$,

the other solutions of (3) can be obtained from the relations

$$\alpha_{n+1} = 72 f_n + \frac{162 g_n}{\sqrt{5}} \qquad (6)$$

$$Y_{n+1} = 162 f_n + 72\sqrt{5 g_n} \qquad (7)$$

Taking positive sign on the R.H.S of (2) and using (4), (6) & (7), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows,

$$x_{n+1} = \frac{1}{2}(3Y_{n+1} - 18 \pm \alpha_{n+1}) \qquad (8)$$

$$y_{n+1} = \frac{1}{5}(3Y_{n+1} + 54) \qquad (9)$$

The recurrence relations satisfied by x_{n+1} , y_{n+1} are respectively

$$x_{n+5} - 322 x_{s+3} + x_{n+1} = -4608$$
$$y_{n+5} - 322 y_{n+3} + y_{n+1} = -3456$$

A few numerical examples are presented in the table below.

n	x_{n+1}	y_{n+1}
0	3042	1170
2	977202	373266
4	314651394	120187026

A few interesting relations among the solutions are presented below.

1) x_{n+1} & y_{n+1} are always even

2) $x_{n+1} \equiv 0 \pmod 2$ (6)

3) $y_{n+1} \equiv 0 \pmod 2$

4) $\frac{3}{81}\left[945 y_{2n+2} - 360 x_{2n+2} - 7614\right] + 12$ is a nasty number.

5) $\frac{1}{162}\left[945 y_{2n+2} - 360 x_{2n+2} - 7614\right] + 2$ is a quadratic integer.

6) $$\frac{1}{162}[945y_{3n+3}-360x_{3n+3}-7614]+3\left[\frac{1}{162}(945y_{n+1}-360x_{n+1}-7614)\right]$$

is a cubic integer.

7) $$\frac{1}{162}\{\begin{pmatrix}945y_{3n+3}-360x_{3n+3}-7614+\\2835y_{n+1}-1080x_{n+2}-22842\end{pmatrix}-\frac{1}{131220}[810x_{n+1}-2115y_{n+1}+17010]^2(945y_{n+1}-360x_{n+1}-7614)\}$$
$$=\frac{2}{84}[945y_{n+1}-360x_{n+1}-7614]$$

8) $y_{n+3}-144x_{n+1}+55y_{n+1}=-432$

9) $y_{n+5}-46368x_{n+1}+17711y_{n+1}=-142560$

10) $x_{n+3}-377x_{n+1}+144y_{n+1}=-1152$

11) $322y_{n+3}-y_{n+5}-y_{n+1}=3456$

12) $144x_{n+1}-17711y_{n+3}+55y_{n+5}=-189648$

13) $377y_{n+3}-144x_{n+3}-y_{n+1}=3024$

14) $144y_{n+3}-55x_{n+3}-x_{n+1}=1152$

15) $144y_{n+5}-17x711x_{n+3}+55x_{n+1}=-125568$

16) $377y_{n+5}-46368x_{n+3}+55y_{n+1}=-329184$

17) $$\frac{1}{162}[945y_{2n+2}-360x_{2n+2}-7614]+2=\left[\frac{1}{162}(945y_{2n+2}-360x_{2n+2}-752)\right]^2$$

18) $$\frac{1}{162}[945y_{3n+3}-360x_{3n+3}-752]+3\left[\frac{1}{162}(945y_{n+1}-360x_{n+1}-752)\right]=\left[\frac{1}{162}(945y_{n+1}-360x_{n+1}-752)\right]^3$$

Remarkable observations:

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the hyperbola.

$$131220U^2-V^2=524880$$

where

$$U=\frac{1}{162}(945y_{n+1}-360x_{n+1}-7614)$$

$$V=810x_{n+1}-2115y_{n+1}+17010$$

$$103680\,U_1^2-V_1^2=414720$$

where

$$U_1=\frac{1}{162}(945y_{n+1}-360x_{n+1}-7614)$$

$$V_1=\frac{1}{144}(5y_{n+3}-1605y_{n+1}+17280)$$

2) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the parabola.

$$N^2=162M-52488$$

Where

$$M=945y_{2n+2}-360x_{2n+2}-7614$$

$$N=810x_{n+1}-2115y_{n+1}+17010$$

$$N_1^2=640M_1-207360$$

Where

$$M = 945y_{2n+2} - 360x_{2n+2} - 7614$$

$$N_1 = -1605y_{n+1} + 5y_{n+3} + 17280$$

CONCLUSION :

In this paper , we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude , one may search for other choices of solutions to the considered binary equation and further , quadratic equations with multi-variables.

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