Geometric construction of D = 4, N = 1 pure supergravity

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Abstract - ***We consider in this paper the technical construction of D = 4, N = 1 pure supergravity in a geometrical way, using superforms in superspace as extension of spinor-tensor calculus. Starting by the important concepts of supersymmetry, superspace and rheonomic principle, we will focus on pure D = 4, N = 1 supergravity and in a technically rigorous geometrical way the structure of the theory will be built.***

Key Words: *Supersymmetry, Superspace, Supergravity, Rheonomy, Differential Geometry, Spinor-Tensor Calculus, Forms Calculus.*

1. INTRODUCTION

Supersymmetry is a modern symmetry with interesting features at pure and applicative level, which represents different properties with respect to the ordinary symmetries of physics. N = 1 supersymmetry is the simplest version for it, with N the number of supersymmetry generators. If we consider a local version of the supersymmetric theory, it necessarily includes the gravity and is said "supergravity". Supersymmetry has the important feature to "keep natural" the standard model, i.e. to adjust with accuracy the "input" parameters of this model, concerning strong, weak and electromagnetic interactions [1].

Supergravity models have a higher predictive power with respect to those based on global supersymmetry; they allow to solve problems such as the "gauge hierarchy" of standard model, permitting a "mass splitting" among fermions and bosons of the same multiplet, in accordance with phenomenological results. This fact opened the way to application of N = 1 spontaneously broken supergravity for the description of the particles phenomenology at low energies.

Field theories including gravity are not renormalizable; this is the case of N = 1 supergravity too. The existence of supersymmetry leads to cancellations of "infinites" of theory, but only at the first perturbative order, not in general. But considering supergravity as the "effective theory" of a superstring theory, not as "fundamental theory", it is possible to shown that the effective theories derived from superstring theories are supergravity theories; N = 1, D = 4 supergravity theory can be considered as the compactification of the heterotic string theory [2,3].

2. PURE D = 4, N = 1 SUPERGRAVITY

We start by the action of Einstein-Cartan:

$$A=\int_{M_4} R^{ab}(\omega)\wedge V^c \wedge V^d\,\varepsilon_{abcd}\ ; \tag{1}$$

M_4 is a 4- dimensional Riemannian manifold, R^{ab} is the 2-form of curvature:

$$R^{ab}=d\omega^{ab}-\omega^a{}_c\wedge\omega^{cb}\ , \tag{2}$$

and V^a is the vierbein. The action (1) is equivalent to the action of gravity, written in the tensor formalism:

$$R^{ab}\wedge V^c\wedge V^d\,\varepsilon_{abcd}=-4\,R^{ij}{}_{ij}\,\det V\,d^4x\ . \tag{3}$$

$R^{ij}{}_{ij}=R^{\mu\nu}{}_{\mu\nu}=R$ is the scalar of curvature and $\det V=\sqrt{-g}=\sqrt{-\det g_{\mu\nu}}$, with Latin letters for flat indices and Greek letters for curved indices. The two involved gauge fields are the spin connection ω^{ab} and the vierbein V^a :

$$\omega^{ab}=\omega^{ab}_{\mu}\,dx^{\mu}\ , \tag{4}$$

$$V^a=V^a_{\mu}\,dx^{\mu}\ . \tag{5}$$

In the first order formalism, both gauge fields are treated as independent. The quantity $\{V^a,\ \omega^{ab}\}$ constitutes a multiplet in the adjoint representation of the Poincarè group; from the 2-form in the Poincarè Lie algebra-valued curvature, we find:

$$R^{ab}=d\omega^{ab}-\omega^a{}_c\wedge\omega^{cb}\ ; \tag{6}$$

$$R^a=dV^a-\omega^a{}_b\wedge V^b\ \equiv\ \mathscr{D}V^a. \tag{7}$$

i.e. the Lorentz algebra-valued curvature is the field strength of the spin connection and the vector-valued curvature, or torsion, is the field strength of the vierbein field. The Einstein-Cartan action is invariant under general

transformations of coordinates generated by Lie derivatives.
The variation of the action (1) with respect to V^a and ω^{ab} brings to the Einstein equation of pure gravity and the condition of the torsion respectively:

$$R^{al}_{\ bl} - \frac{1}{2}\delta^a_b R = 0\,; \tag{8}$$

$$R^c \wedge V^d \varepsilon_{abcd} = 0 \ \rightarrow \ R^c = 0\,. \tag{9}$$

The next step consists in the coupling of the graviton with its supersymmetric partner, or "superpartner", i.e. the "gravitino". It is described by the spin 3/2 Rarita-Schwinger field:

$$\psi = \psi^\alpha Q_\alpha = \psi^\alpha_{\ \mu}(x)\, dx^\mu Q_\alpha\,. \tag{10}$$

$\psi^\alpha_{\ \mu}(x)$ describes a massless particle with spin 3/2 in spacetime and Q_α is the supersymmetry generator; ψ_μ satisfies the Majorana condition $\psi^+ \gamma^0 = \psi^t C$.
The equation of motion of gravitino is:

$$\varepsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma = 0\,. \tag{11}$$

The complete action describing the coupling "graviton + gravitino" is:

$$A = \int_{M_4} -4R\sqrt{-g}\, d^4x + 4\,\overline{\psi}_\mu\, \gamma_5\, \gamma_a$$

$$\mathscr{D}_\nu\, \psi_\rho\, V^a_\lambda\, \varepsilon^{\mu\nu\rho\lambda}\, d^4x\,. \tag{12}$$

If we use forms, the action becomes:

$$A = \int_{M_4} R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} + 4\,\overline{\psi} \wedge \gamma_5\, \gamma_a\, \mathscr{D}\psi \wedge V^a\,. \tag{13}$$

The third independent field is therefore:

$$\psi = \psi_\mu(x)\, dx^\mu\,. \tag{14}$$

3. THE ROLE OF SUPERSPACE

It is favorable in the context of formalism generalization, and for giving geometric meaning to the supersymmetry transformations, to consider the three previously introduced spacetime fields $V^a_{\ \mu}$, ψ_μ, $\omega^{ab}_{\ \ \mu}$ as 1-forms in superspace. Such procedure allows to interpret the supersymmetry transformations as Lie derivatives in superspace, where the 1-forms (V^a, Ψ) can be considered as a single object E^a = (V^a, Ψ), said "supervielbein". Supergravity can be "naturally" interpreted as a theory in superspace.
We define the curvatures:

$$R^{ab} = d\omega^{ab} - \omega^a_{\ c} \wedge \omega^{cb} \equiv \mathscr{R}^{ab}, \tag{15}$$

$$R^a = \mathscr{D} V^a - \frac{i}{2}\overline{\psi} \wedge \gamma^a \psi\,, \tag{16}$$

$$\rho = \mathscr{D}\psi\,. \tag{17}$$

Now ω^{ab}, V^a, Ψ are 1-forms in superspace, and R^{ab}, R^a and ρ the corresponding curvatures. It is possible to use a compact notation:

$$R^A = d\mu^A + \frac{1}{2} C^A_{BC}\, \mu^B \wedge \mu^C\,, \tag{18}$$

with $R^A = \left(R^{ab}, R^a, \rho\right)$ [2].

4. ABOUT THE RHEONOMIC PRINCIPLE

The fields ω^{ab}, V^a, Ψ, introduced for the spacetime description of supergravity, enter in the structure equations of superspace. The 1-forms $\mu^A = \left(\omega^{ab}, V^a, \psi\right)$ are defined on a fiber bundle such as $\tilde{G} = \tilde{G}\left(M_{4/4}, SO(1,3)\right)$, the fields of supergravity in the "standard components approach" are defined only on spacetime M_4. The identification of 1-forms with the supergravity fields requires that the spacetime fields $\omega^{ab}_{\ \ \mu}(x)$, $V^a_{\ \mu}(x)$, $\psi^\alpha_{\ \mu}(x)$ are interpreted as "boundary spacetime values" of the superspace superfields:

$$V^a = V^a_\mu(x)\, dx^\mu \quad \rightarrow \quad V^a(x) = V^a(x,\theta)\big|_{\substack{\theta=0 \\ d\theta=0}}\,; \tag{19}$$

$$\psi = \psi_\mu(x)\, dx^\mu \quad \rightarrow \quad \psi(x) = \psi(x,\theta)\big|_{\substack{\theta=0 \\ d\theta=0}}\,; \tag{20}$$

$$\omega^{ab} = \omega^{ab}_\mu(x)\, dx^\mu \quad \rightarrow \quad \omega^{ab}(x) = \omega^{ab}(x,\theta)\big|_{\substack{\theta=0 \\ d\theta=0}}\,. \tag{21}$$

They are the values in $\theta^\alpha = 0$ of restriction on the cotangent bosonic plane of the corresponding 1-forms in superspace. The "mapping":

$$\text{rh:} \begin{Bmatrix} V^a(x) \rightarrow V^a(x,\theta) \\ \psi(x) \rightarrow \psi(x,\theta) \\ \omega^{ab}(x) \rightarrow \omega^{ab}(x,\theta) \end{Bmatrix} \tag{22-24}$$

is said “mapping of rheonomic extension”.
The concept of rheonomy is introduced by assuming that the “outer” components $R^{A}{}_{\alpha L}$ can be algebraically expressed in terms of the “inner” (or purely spacetime) components $R^{A}{}_{\mu\nu}$:

$$R^{A}{}_{\alpha L} = C^{A/\mu\nu}_{\alpha L/B} R^{B}_{\mu\nu} . \tag{25}$$

In Eq. (25) $C^{A/\mu\nu}_{\alpha L/B}$ are constant tensors, μ and ν are indices of spacetime bosonic coordinates, α is a spinorial index associated to coordinate θ^{α}, $L=(\alpha,\mu)$, A e B are indices of Lie superalgebra. In presence of rheonomic constraints, the physical content of a superspace theory is entirely determined by a purely spacetime description. Therefore a supersymmetry transformation can be identified with the Lie derivative l_{ε} in presence of rheonomic constraints.
In relation to the action, it is normally written as:

$$S = \int_{\Omega} \mathcal{L}(\varphi), \tag{26}$$

with Ω n-dimensional manifold, $\mathcal{L}$ scalar density, i.e. a n-form created with p-forms on Ω. The action is therefore a functional of the fields configurations only. We can generalize the action (26) considering as Lagrangians $\mathcal{L}(\varphi)$ forms with degree $D < n$:

$$S = S[\varphi, M_D] = \int_{M_D} \mathcal{L}(\varphi), \tag{27}$$

with M_D a D-dimensional sub-manifold of Ω.
In this way the action becomes a functional of the field configurations $\varphi(x)$ and of the surface M_D. Minimizing S with respect to variations of fields and of the surface, we obtain the classical equations of motion. The difficulties in resolution of such equations can be overcome considering the fields $\{\varphi_i\}$ as a set of exterior forms of various degrees p and the Lagrangian obtained from $\{\varphi_i\}$ using only the “diffeomorphism-invariant” operations of exterior algebra, i.e. the exterior derivative “d”: $\varphi \to d\varphi$, the wedge product “$\wedge$”: $(\varphi_1 , \varphi_2) \to \varphi_1 \wedge \varphi_2$ and excluding the Hodge operator of forms dualisation “ * “.
Following this way, the deformation of the surface M_D may be compensated by a diffeomorphism in the superspace of fields $\{\varphi_i\}$. This implies that the full set of variational equations associated with the action is given by the usual equations of motion obtained by varying action with respect to the fields “on a fixed surface”. The equations are valid on the whole superspace and the action is “geometric”, i.e. built with exterior forms, using “ d ” and “ $\wedge$ ” and excluding “ * “ [4].
The extended action is:

$$A^{D=4,N=1}_{extended} = \int_{M_4 \subset R^{4/4}} R^{ab} \wedge V^{c} \wedge V^{d} \varepsilon_{abcd} \; +$$

$$+4\,\overline{\psi} \wedge \gamma_5\, \gamma_a\, \rho \wedge V^{a} , \tag{28}$$

where $R^{4/4}$ is a superspace manifold and M_4 is a four dimensional bosonic surface, identifiable (for example) at $\theta = 0$ with a spacetime surface. The equations of motion have the same form of those in the spacetime approach, because the Lagrangian is geometric, but now they hold on the whole superspace $R^{4/4}$. These equations are 3-forms in $R^{4/4}$; it is possible to expand them in a complete basis of 3-forms and analyze them [4,5]. The $V \wedge V \wedge V$ projection gives the propagation equations for the spacetime components R^{ab}_{cd}, R^{a}_{bc} e ρ_{ab}, valid on whole $R^{4/4}$. The restriction on M^4 brings to the ordinary equations on spacetime. The equations of motion in superspace imply also the rheonomic constraints, which are differential relations satisfying the integrability conditions $d^2 = 0$. The examination of the $\psi \wedge V \wedge V$ content of Bianchi torsion gives the spacetime equation of gravitino; the Bianchi-Riemann identity gives the spacetime Einstein equation and the symmetry property of the Ricci tensor.

5. INFORMATIONS ON THE THEORY THROUGH THE RHEONOMIC PRINCIPLE AND BIANCHI IDENTITIES

The Bianchi identities imply in general the equations of motion of the theory. There is also another line of approach for the building of a theory of supergravity, i.e. not starting with the Lagrangian, but assuming the rheonomic principle as the basic starting point of the theory. The rheonomic principle in the usual form is assumed, i.e. the outer components of the 2-forms of curvature in superspace must be expresses “only” in terms of the inner components. The parameterization of the curvatures in superspace, in addition to satisfy the rheonomic principle, must satisfy also the Lorentz covariance and a scaling invariance, which is below defined. The reason lies in the fact that the definition of the curvatures and the corresponding Bianchi identities are Lorentz-covariant and are also invariant with respect to the following scaling transformation:

$$\omega^{ab} \to \omega^{ab} , \quad (29) \qquad R^{ab} \to R^{ab} , \quad (30)$$

$$V^{a} \to wV^{a} , \quad (31) \qquad R^{a} \to wR^{a} , \quad (32)$$

$$\psi \to \sqrt{w}\,\psi , \quad (33) \qquad \rho \to \sqrt{w}\,\rho , \quad (34)$$

where "*w*" is a constant non-zero parameter said "scaling factor". All this allows to write the most general parameterization of the curvatures in superspace:

$$R^{ab}=R^{ab}_{cd}V^c\wedge V^d+\bar{\theta}^{ab}{}_c\,\psi\wedge V^c+\bar{\psi}\wedge K^{ab}\psi\,, \tag{35}$$

$$R^a=R^a{}_{bc}V^b\wedge V^c+\bar{\theta}^a{}_c\,\psi\wedge V^c+\bar{\psi}\wedge K^a\psi\,, \tag{36}$$

$$\rho=\rho_{ab}V^a\wedge V^b+H_c\,\psi\wedge V^c+\Omega_{\alpha\beta}\psi^\alpha\wedge\psi^\beta\,, \tag{37}$$

Due to the rheonomic principle, tensors $\theta^{ab}{}_c$, K^{ab}, H_c, $\Omega_{\alpha\beta}$ must be built with the "inner" curvatures, i.e. in terms of the spacetime curvatures $R^{ab}{}_{mn}$, $R^a{}_{mn}$ e ρ_{mn}. If it is assumed that the 2-form R^a is identically zero, it follows that $R^a{}_{mn}=0$; with a redefinition of the spin connection it is always possible to put the supertorsion R^a equal to zero. By Eqs (29-37) it follows that the scale dimensions of the various introduced tensors are:

$[\theta^{ab}{}_c]$ = - 3/2 , (38) $[H^c]$ = -1, (39)

$[K^{ab}]$ = -1 , (40) $[\Omega_{\alpha\beta}]$ = - 1/2. (41)

Introducing the parametrizations (35-37) in the Bianchi identity of torsion and considering the $\psi\wedge V\wedge V$ projection, with $R^a=0$, we obtain:

$$\bar{\theta}^{ab}{}_c\,\psi\wedge V_b\wedge V^c+\ i\bar{\psi}\wedge\gamma^a\rho_{bc}V^b\wedge V^c=0\,, \tag{42}$$

or, equivalently:

$$\frac{1}{2}(\bar{\theta}_{ab/c}-\bar{\theta}_{ac/b})\psi=i\bar{\rho}_{bc}\gamma_a\psi\,. \tag{43}$$

Solving this last, we find:

$$\bar{\theta}_{ab/c}=2i\bar{\rho}_{c[a}\gamma_{b]}-i\bar{\rho}_{ab}\gamma_c\,. \tag{44}$$

In this way the curvatures of superspace are fixed as follows:

$$R^{ab}=R^{ab}_{cd}V^c\wedge V^d+(2i\bar{\rho}_{c[a}\gamma_{b]}-$$

$$-i\bar{\rho}_{ab}\gamma_c)\,\psi\wedge V^c\,, \tag{45}$$

$$R^a=0\,, \tag{46}$$

$$\rho=\rho_{ab}V^a\wedge V^b\,. \tag{47}$$

The parametrization (45) is different from that found by solving the equations of motion in superspace. This is not in contrast with what previously stated, because there is a difference only in the left-hand side of the spacetime equation of gravitino; but $\gamma^a\rho_{ab}=0$, as it can be calculated, therefore they are equivalent.

We can get by Bianchi identities also the equations of motion on spacetime. Indeed, if we consider the $\psi\wedge\psi\wedge V$ sector of the Bianchi identity of gravitino and use Eqs (45-47), we find:

$$\rho_{ab}\bar{\psi}\wedge\gamma^a\psi\wedge V^b+\frac{1}{4}\gamma^{ab}\psi\wedge\bar{\psi}\times$$

$$\times(\gamma_c\rho_{ab}-2\gamma_a\rho_{bc})\wedge V^c=0\,. \tag{48}$$

Using the Fierz decomposition:

$$\psi\wedge\bar{\psi}=\frac{1}{4}\gamma^a\psi\wedge\gamma_a\psi-\frac{1}{8}\gamma^{ab}\psi\wedge\gamma_{ab}\psi\,, \tag{49}$$

we obtain two equations for the coefficients of $\psi\wedge\gamma_a\psi\wedge V^b$ and $\psi\wedge\gamma_{pq}\psi\wedge V^c$ respectively:

$$\frac{1}{8}\rho_{ab}+\frac{i}{16}\gamma_5\varepsilon_{ab}{}^{cd}\rho_{cd}+\frac{1}{16}\delta_{ab}\gamma^{pq}\rho_{pq}-$$

$$-\frac{1}{4}\gamma_{[a}\gamma^p\rho_{b]p}=0, \tag{50}$$

$$-\gamma_{pq}\gamma^a\rho_a{}^c-4\gamma^b{}_{[q}\gamma^c\rho_{p]b}=0. \tag{51}$$

The left members of Eqs (50, 51) are proportional to the left-hand side of the field equation of gravitino; Therefore we find:

$$\gamma_5\gamma_r\varepsilon_p{}^{rst}\rho_{st}=\gamma^a\rho_{ap}=0\,. \tag{52}$$

For finding the Einstein equation we can perform a Lie derivative of Eq. (52); it is however equivalent, even more convenient, to use:

$$8\gamma_5\gamma_m\rho\wedge V^m-4\gamma_5\gamma_m\psi\wedge R^m=0\,. \tag{53}$$

with $R^a=0$ and differentiate it. We obtain:

$$\mathscr{D}(\gamma_5\gamma_m\rho\wedge V^m)=\gamma_5\gamma_m\mathscr{D}\rho\wedge$$

$$\wedge V^m+\frac{i}{2}\gamma_5\gamma_m\rho\wedge\bar{\psi}\wedge\gamma^m\psi=0\,. \tag{54}$$

Using the Bianchi identity of gravitino, Eq. (54) becomes:

$$\gamma_5 \gamma_m \left(-\frac{1}{4} R^{ab} \gamma_{ab} \wedge \psi \wedge V^m + \frac{i}{2} \rho \wedge \overline{\psi} \wedge \gamma^m \psi\right) = 0 \cdot \quad (55)$$

In $\psi \wedge V \wedge V \wedge V$ sector we find:

$$\gamma_m \gamma_{ab}\, \psi \wedge R^{ab}{}_{cd}\, V^c \wedge V^d \wedge V^m = 0, \quad (56)$$

i.e.:

$$i \gamma_5 \gamma^t\, \psi\, \varepsilon_{mabt}\, R^{ab}{}_{cd}\, V^c \wedge V^d \wedge V^m +$$

$$+\ 2\gamma_b\, \psi \wedge R^{mb}{}_{cd}\, V^c \wedge V^d \wedge V_m = 0 \cdot \quad (57)$$

The second term of Eq. (57) is zero, for the cyclic identity of the Riemann tensor, valid when $R^a = 0$. The first term gives the Einstein equation:

$$R^{ma}{}_{mb} - \frac{1}{2} \delta^a_b\, R^{mn}{}_{mn} = 0, \quad (58)$$

having used the equality:

$$V^c \wedge V^d \wedge V^m = \varepsilon^{cdmf}\, \Omega_f\, . \quad (59)$$

6. CONCLUSIONS

In this paper we have considered the rigorous construction of D = 4, N = 1 pure supergravity in a geometrical way, i.e. using superforms in superspace. We underlined fundamental concepts, such as that of supersymmetry, superspace and rheonomic principle, for focusing then the attention on pure D = 4, N = 1 supergravity. The rheonomic principle, implemented in the Bianchi identities, togheter with the Lorentz gauge invariance and the scaling property, have the same "on-shell" content of the equations derived from the principle of extended action. We saw that two alternatives for building a classical theory of supergravity are possible:

a) the use of the principle of extended action and the rules related to its construction;

b) the use of the Bianchi identities, with the rheonomic principle, the Lorentz gauge invariance and the scaling property.

In both cases it is fundamental the use of the rheonomic principle [6-11].

REFERENCES

[1] P. Di Sia, *Supergravità nel superspazio: panoramica generale e analisi tecnica,* Italy: Aracne Editrice, 2014.

[2] P. Di Sia, "About the importance of supersymmetry and superspace for supergravity", *International Journal of Innovative Science, Engineering and Technology* (IJISET), vol. 2(4), pp. 495-500, April 2015.

[3] D. Z. Freedman, A. Van Proeyen, *Supergravity*, UK: Cambridge University Press, 2012.

[4] P. Di Sia, "Rheonomic principle, Bianchi identities and supergravity", *International Journal of Innovative Science, Engineering and Technology* (IJISET), vol. 2(5), pp. 615-619, May 2015.

[5] P. Di Sia, "Relevant tools in building supergravity theories", *Integrated Journal of British*, in press, 2015.

[6] P. Di Sia, "Kähler manifolds as target space of supergravity theories: characteristics for D = 4, N = 1 theory", *International Journal of Current Research*, vol. 7(1), pp. 11971-11980, Jan 2015.

[7] H. Nastase, *Introduction to supergravity*, http://arxiv.org/pdf/1112.3502.pdf, 2012.

[8] M. B. Green, J. H. Schwarz, E. Witten, *Superstring Theory: 25th Anniversary Edition*, Cambridge Monographs on Mathematical Physics, UK: Cambridge University Press, vol. 1, 2012.

[9] P. Di Sia, "Extreme Physics and Informational / Computational Limits", *Journal of Physics: Conference Series*, N. 306, p. 012067 (8 pp.), 2011.

[10] P. Di Sia, "Exciting Peculiarities of the Extreme Physics", *Journal of Physics: Conference Series*, N. 442(1), p. 012068 (6 pp.), 2013.

[11] C. J. Isham, *Modern Differential Geometry for Physicists*, Singapore: World Scientific Pub Co Inc, 1999.

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