

Enumeration of size of a T- graphs

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Abstract- A Graph (G, E_T) be a finite T-Graph, i.e., the graph whose vertex set be the power set : $P(X)$ of a non empty set and edge set $E_T = \{(u, v): u \in T, u \neq v \& u^c \cap v \in T\}$ where T be a topology on a non-empty finite set X . In this article, we associate a graph to a topology T , called finite T -Graph of the topology T . Our goal is to find the enumeration of edges between nodes of the finite T-Graphs and its size.

Key Words- Topology, size of a graph, edge.

1. Introduction

According to Dongseok Kim *et.al* [7] and Topology on a finite set is often known as finite topology and in their work there is one-to-one correspondence between the topologies on a non-empty set and its corresponding underlying digraphs. Finite topology also plays a key role in the theory of image analysis[9,10], the structures of molecular[8], geometries of finite sets[2] and digital topology.

There are two approaches in the study of combined work on topology and graph theory. One is from graph theory to topology and another one is from topology to graph theory [1]. The pioneer work of Nagaratnamaiah *et.al.*, [3,4,5,6], use from topology to graph theory which shows the way to us for doing this work. One of research from topology to graph theory is to find the enumeration of node degrees. In the present article, we mainly focus on newly defined associated graphs with topology and the enumeration of edges between nodes of the finite T-Graphs and its size.

The outline of this article is as follows. First, we will provide precise definitions and some general formulae in Section 2. In final Section, we have our contribution propositions.

2. Preliminaries

For a finite non-empty set X , let T be a topology on X . the graph associated with this topology T is (G, E_T) whose vertex set is the power set $P(X)$ of X and edge set be the $E_T = \{(u, v): u \in T, u \neq v \& u^c \cap v \in T\}$. In any graph, the degree of the node is the number of edges incident on it. The number of combinations of n objects taken r at a time $\binom{n}{r}$.

3. Our contribution propositions

In this section, there are interesting proposition for the enumeration of edges between nodes of the finite T-Graphs and its size.

Proposition 3.1. In any graph (G, E_T) , the maximum number of edges between any two nodes u, v of (G, E_T) is two.

There are four possible cases of any two nodes u, v of a graph (G, E_T) .

case 1: $u, v \in T$

According to our definition, there is edge between these two nodes.

That is the number of edges between u, v is zero.

Case 2: $u \in T, v \notin T$.

Sub case 1: $u^c \in T$

Since T is a topology, so either $v \cap u^c \in T$ or $v \cap u^c \notin T \Rightarrow \langle u, v \rangle \in E_T$ or $\langle u, v \rangle \notin E_T$ so, the nodes u, v has either one edge or no edge in this sub case.

Sub case 2: $u^c \notin T$.

Since T is a topology, so either $v \cap u^c \in T$ or $v \cap u^c \notin T \Rightarrow \langle u, v \rangle \in E_T$ or $\langle u, v \rangle \notin E_T$ so, the nodes u, v has either one edge or no edge in this sub case.

Thus, in this case there is atleast one edge between any two nodes.

Case 3: $u \in T, u \notin T$.

Sub case 1: $v^c \in T$

Since T is a topology, so either $u \cap v^c \in T$ or $u \cap v^c \notin T$

$\Rightarrow \langle u, v \rangle \in E_T$ or $\langle u, v \rangle \notin E_T$ so, the nodes u, v has either one edge or no edge in this sub case.

Sub case 2: $v^c \notin T$.

Since T is a topology, so either $u \cap v^c \in T$ or $u \cap v^c \notin T$

$\Rightarrow \langle u, v \rangle \in E_T$ or $\langle u, v \rangle \notin E_T$ so, the nodes u, v has either one edge or no edge in this sub case.

Thus, in this case there is atleast one edge between any two nodes.

Case 4: $\emptyset \in T, u \in T$.

Sub case 1: $u^c \in T$ and $v^c \in T$

Since T is a topology, so either $v \cap u^c \in T$ or $v \cap u^c \notin T$

\Rightarrow the nodes u, v have two edges

Sub case 2: $u^c \notin T$ and $v^c \in T$

Since T is a topology, so either $u \cap v^c \in T$ or $u \cap v^c \notin T$

$\Rightarrow \langle u, v \rangle \in E_T$ or $\langle u, v \rangle \notin E_T$ so, the nodes u, v has either one edge or no edge in this sub case.

Sub case 3: $u^c \in T$ and $v^c \notin T$.

Since T is a topology, so either $v \cap u^c \in T$ or $v \cap u^c \notin T$

$\Rightarrow \langle u, v \rangle \in E_T$ or $\langle u, v \rangle \notin E_T$ so, the nodes u, v has either one edge or no edge in this sub case.

Sub case 4: $u^c \notin T$ and $v^c \notin T$.

Since T is a topology, so $v \cap u^c \in T$ or $v \cap u^c \notin T$ or $u \cap v^c \in T$ or $u \cap v^c \notin T$

$\Rightarrow \langle u, v \rangle \in E_T$ atmost twice

Finally, in this case the nodes u, v have atmost two edges.

Thus, in this case there is atmost two edge between any two nodes.

Hence, from all these cases, we have that the maximum number of edges between two any two nodes u, v of (G, E_T) is two.

Proposition 3.2. In any graph (G, E_T) , the number of edges between the nodes \emptyset, u is zero if and only if $u \in T$.

Suppose the number of edges between the nodes \emptyset and u is zero.

To show that $u \in T$. This is proved in a contradiction way. Assume that $u \notin T$.

We know that $\emptyset \in T, \emptyset^c = X$ and $X \cap u = u, \forall u \subseteq X$. Thus $\emptyset^c \cap u = X \cap u = u \in T$ so, $\langle \emptyset, u \rangle \in E_T$

Also $u^c \cap \emptyset = \emptyset \in T \Rightarrow \langle u, \emptyset \rangle \in E_T$. Therefore, the nodes \emptyset, u have two edges. Which is a contradicts the hypothesis. So our assumption is wrong. Thus $u \in T$ if the number of edges between the nodes \emptyset and u is zero.

Conversely, suppose $u \in T$.

To show that the number of edges between the nodes \emptyset and u is zero. This is proved in a contraction way.

Assume that the number of edges between the nodes \emptyset and u is not zero. \Rightarrow there atleast one edge between the nodes \emptyset and u . $\Rightarrow \emptyset^c \cap u$ or $u^c \cap \emptyset \in T \Rightarrow u$ or $\emptyset \in T \Rightarrow u \in T$. Therefore, number of edges between the nodes \emptyset and u is two. $\Rightarrow \emptyset^c \cap u, u^c \cap \emptyset \in T \Rightarrow u, \emptyset \in T \Rightarrow u \in T$. Which is a contradiction to our hypothesis. So, our assumption is wrong. Therefore the number of edges between the nodes \emptyset and u is one. Hence, the number of edges between the node \emptyset and node u is one if and only if $u \in T$.

Proposition 3.3. In any graph (G, E_T) , the number of edges between the node \emptyset and node u is two if and only if $u \notin T$.

Suppose the number of edges between the nodes \emptyset and u is two.

To show that $u \notin T$. This is proved in a contradiction way. Assume that $u \in T$.

We know that $\emptyset \in T, \emptyset^c = X$ and $X \cap u = u, \forall u \subseteq X$. Thus $\emptyset^c \cap u = X \cap u = u \in T$ so, $\langle \emptyset, u \rangle \in E_T$

Also $u^c \cap \emptyset = \emptyset \in T \Rightarrow \langle u, \emptyset \rangle \in E_T$. Therefore, the nodes \emptyset, u have one edge. Which is a contradicts the hypothesis. So our assumption is wrong. Thus $u \notin T$ if the number of edges between the nodes \emptyset and u is one. Conversely, suppose $u \notin T$.

To show that the number of edges between the nodes \emptyset and u is two. This is proved in a contraction way.

Assume that the number of edges between the nodes \emptyset and u is not two. Therefore, number of edges between the nodes \emptyset and u is one. $\Rightarrow \emptyset^c \cap u \in T \Rightarrow u \in T \Rightarrow u \in T$. Which is a contradiction to our hypothesis.

So, our assumption is wrong. Therefore the number of edges between the nodes \emptyset and u is one.

Hence, the number of edges between the node \emptyset and node u is one if and only if $u \in T$.

- i.
- ii.

Proposition 3.4. Let A, B are two disjoint members of a topology T such that A^c, B^c are also belongs to T on a non-empty set X . The number of edges between these two nodes is two but converse need not be true.

By the hypothesis $A \cap B = \emptyset$, so according to set theory $A^c \subseteq B, B^c \subseteq A$.

$$\Rightarrow A^c \cap B = A^c \in T \quad \text{and} \quad B^c \cap A = B^c \in T \Rightarrow (A, B), (B, A) \in T.$$

Therefore, the nodes A, B have two edges between them.

Converse need not be true:

Consider a topology $T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ on a set $= \{a, b, c\}$.

Let $A = \{a\}$ and $B = \{a, b\}$

Clearly $A \cap B \neq \emptyset$ but $A^c \cap B = \{b\} \in T \Rightarrow (A, B) \in E_T$

And also $B^c \cap A = \emptyset \in T \Rightarrow (B, A) \in E_T$

Thus the two nodes A, B have two edges even though the hypothesis of this theorem fails.

Proposition 3.5. Let u be an arbitrary node of the graph (G, E_T) . Then the node X and u have atleast one edge.

According to definition of topology, X is a member of T .

According to set theory and topology, $X^c = \emptyset \in T$ and $\emptyset \cap u = \emptyset$. Therefore, $X^c \cap u = \emptyset \in T, \forall u \in T \Rightarrow (X, u) \in E_T$

That is, the nodes X, u have an edge.

Proposition 3.6. Let u be an arbitrary node of the graph (G, E_T) . Then the node X and u have two edges if and only if $u, u^c \in T$.

suppose the nodes the X and u have two edges in (G, E_T) . According to our definition, $u, X^c \cap u, u^c \cap X \in T \Rightarrow u, \emptyset, u^c \in T$ ($\because X^c \cap u = \emptyset, u^c \cap X = u^c$). Therefore, u, u^c are members of T if the nodes X and u have two edges.

Conversely, Suppose u, u^c are members of T . To show that the nodes X and u have two edges. This is proved in a contradiction way. Assume that the nodes X and u have no two edges. By applying above theorem; the nodes X and u have one edges.

There is only one of the following two possibilities :

$$X \in T, X^c \cap u \in T$$

$$u \in T, u^c \cap X \in T$$

That is either $X, \emptyset \in T$ or $u, u^c \in T$ but not both. This contradicts the hypothesis and definition of topology.

Thus our assumption is wrong. Hence there are exactly two edges between the nodes X and u .

Proposition 3.7. show that in a graph (G, E_T) , the nodes \emptyset, u have two edges $\Leftrightarrow u \in T$.

Suppose the nodes \emptyset, u have two edges. To show that $u \in T$. This is proved in a contradiction way.

Assume that $u \notin T$. Therefore, $\emptyset^c \cap u = X \cap u = u \notin T \Rightarrow (\emptyset, u) \in E_T \Rightarrow$ There is no edge between the nodes \emptyset, u .

Which is a contradiction to our hypothesis. Thus, $u \in T$.

Conversely, suppose $u \in T$. To show that the nodes \emptyset, u have two edges between them. This is proved in a contraction way.

Assume that there is no two edges between the nodes $\emptyset, u \Rightarrow$ There are two possibilities : i.

There is no edge between the nodes \emptyset, u ii. There is one edge between the nodes \emptyset, u .

Case 1: There is no edge between the nodes \emptyset, u . This is only possible $\emptyset, u \in T$. This case is not possible because by the definition of a topology $\emptyset \in T$. Thus our assumption is wrong.

Case 2: There is one edge between the nodes \emptyset, u . So, $\emptyset^c \cap u = u \in T$ ($\because \emptyset \in T$)

\Rightarrow there is no edge between the nodes \emptyset, u . Which is not possible. Thus our assumption is wrong.

Hence, the nodes \emptyset, u have two edges $\Leftrightarrow u \in T$.

Proposition 3.8. In a graph (G, E_T) , the maximum size of the graph is $2^{|X|}(2^{|X|} - 1)$.

The number of nodes in a graph is $2^{|X|}$ because vertex set is equal to power set of X . According to counting principal, the maximum number of edges between these $2^{|X|}$ number of nodes is $2^X C_2$ without considering the pair repeatedly. Using the previous theorem, there exists maximum two edges between two nodes.

Therefore, the maximum number of edges in a graph is $2 \cdot 2^X C_2 = 2^{|X|}(2^{|X|} - 1)$. Hence the maximum size of the graph (G, E_T) is $2^{|X|}(2^{|X|} - 1)$.

Proposition 3.9. In a graph (G, E_T) , the minimum size of the graph $2^{|X|} + 2 \cdot |T| - 4$.

The number of nodes in a graph is $2^{|X|}$ because vertex set is equal to power set of X . By the definition of a topology and according to set theory ; i $\emptyset^c \cap u = u \in T, \forall u \in T$ ii $X^c \cap u = \emptyset \in T, u \subseteq X$ iii $u^c \cap \emptyset = \emptyset \in T, \forall u \in T$. \Rightarrow by using the definition the graph ; i the node X has edge with all nodes of the graph. That is the number of edges from X is $(2^{|X|} - 1)$. ii the node \emptyset has edges with all nodes of the graph which are belongs to the topology. That is the number of edges from the node \emptyset is $(|T| - 1)$. iii the each node of this collection $\{u \in T : u \neq \emptyset \neq X\}$ has edge with the node \emptyset . That is the number of edges from the node \emptyset is $(|T| - 2)$. These are the minimum number of edges exist in every graph (G, E_T) . Therefore the total minimum number of edges in graph (G, E_T) is $(2^{|X|} - 1) + (|T| - 1) + (|T| - 2) = 2^{|X|} + 2 \cdot |T| - 4$. Hence the minimum size of the graph (G, E_T) is $2^{|X|} + 2 \cdot |T| - 4$.

Proposition 3.10. In a graph (G, E_T) , the size of the graph is $2^{|X|}$ if and only if T is a indiscrete topology

suppose the size of the graph is $2^{|X|}$. According to above theorem the minimum size of any graph is $2^{|X|} + 2 \cdot |T| - 4$. Therefore, $2^{|X|} + 2 \cdot |T| - 4 \leq 2^{|X|}$ and Since $2 \leq |T|$, so $2^{|X|} + 2 \cdot |T| - 4 \geq 2^{|X|}$. thus, $2^{|X|} + 2 \cdot |T| - 4 = 2^{|X|} \Rightarrow 2 \cdot |T| - 4 = 0 \Rightarrow 2 \cdot |T| = 4 \Rightarrow |T| = 2 \Rightarrow T$ is a indiscrete topology.

conversely, Suppose T is an indiscrete topology on X . To show that the size of the graph is $2^{|X|}$. T is a indiscrete topology, so $T = \{\emptyset, X\}$. Since $X \in T$, So, $X^c \cap u = \emptyset \in T, \forall u \subseteq X$. Therefore the number of edges from X in a graph (G, E_T) is $2^{|X|} - 1$. Since $\emptyset \in T$, so $\emptyset^c \cap X = X \in T, \emptyset^c \cap u = u \in T \forall X \neq u \in T$. Therefore the number of edges from \emptyset in a graph (G, E_T) is 1. Thus the total number of edges in this graph is $2^{|X|} - 1 + 1 = 2^{|X|}$.

Hence, the size of the graph is $2^{|X|}$ if and only if T is a indiscrete topology.

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